Fourier-Mukai Transforms and Stability Conditions on Abelian Threefolds

Hathurusinghege Dulip Bandara Piyaratne

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Hathurusinghege Dulip Bandara Piyaratne*)

To my parents.

Abstract

Construction of Bridgeland stability conditions on a given Calabi-Yau threefold is an important problem and this thesis realizes the first known examples of such stability conditions. More precisely, we construct a dense family of stability conditions on the derived category of coherent sheaves on a principally polarized abelian threefold X with Picard rank one. In particular, we show that the conjectural construction proposed by Bayer, Macrì and Toda gives rise to Bridgeland stability conditions on X. First we reduce the requirement of the Bogomolov-Gieseker type inequalities to a smaller class of tilt stable objects which are essentially minimal objects of the conjectural stability condition hearts for a given smooth projective threefold. Then we use the Fourier-Mukai theory to prove the strong Bogomolov-Gieseker type inequalities for these minimal objects of X. This is done by showing any Fourier-Mukai transform of X gives an equivalence of abelian categories which are double tilts of coherent sheaves.

Lay Summary

Geometry can be described as the study of shapes and objects. Algebraic Geometry is one of its branches concerned with questions of geometric objects called algebraic varieties that arise as zero sets of polynomials. The classical perspective in the subject is to study geometric properties of the varieties directly. But a more modern approach is to study the varieties indirectly via some algebraic objects associated to them. Sheaf theory is such an algebraic tool which encodes both the local and global information of a variety. The main idea of homological algebraic methods is to embed the objects into a more fitting world of complexes, where less information gets lost. In particular, the theory of derived categories provides an efficient algebraic platform to investigate the "hidden" geometric information of a variety.

The notion of stability appears in various forms in Algebraic Geometry and it is fundamental in the construction of certain parameter spaces called moduli spaces of sheaves. Bridgeland introduced a categorical stability notion motivated by Physics, and his approach can be considered as an abstraction of the usual slope stability for sheaves. Construction of such stability conditions on the derived categories of certain three dimensional varieties called Calabi-Yau threefolds, is an important problem. One of the main aims of this thesis is to obtain the first known examples of such Bridgeland stability conditions. More precisely, we construct a dense family of stability conditions on the derived category of an abelian threefold. The main tool that we use is some interesting symmetries of the derived category, known as the Fourier-Mukai transformations. In particular, we study the stability of sheaves and also of complexes under these transforms in great detail.

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Contents

Notation

- A variety will always refer to an algebraic variety over the field of complex numbers. The dimension of a variety X over the field of complex numbers is denoted by $\dim X$.

- When X, Y are varieties, the map $\mathsf{Swap} : X \times Y \to Y \times X$ is defined by $\mathsf{Swap}(x, y) = (y, x)$.

- When $Y \subset X$ is a closed subscheme of X, the structure sheaf of Y as an object of $\mathrm{Coh}(X)$ is denoted by $\mathcal{O}_Y$. Moreover, the skyscraper sheaf of a closed point $x \in X$ is denoted by $\mathcal{O}_x$, and it is also the structure sheaf of $\{x\} \subset X$.

- Let p_1, p_2 be the projection maps from the product $X \times Y$ to the varieties X, Y respectively. If $E \in D^b(X)$ and $F \in D^b(Y)$ then

$$E \boxtimes F = p_1^*(E) \overset{\mathbf{L}}{\otimes} p_2^*(F) \in D^b(X \times Y).$$

- For a triangulated category $\mathcal{D}$, if $S \subset \mathcal{D}$ is a set of objects of $\mathcal{D}$ then $\langle S \rangle$ denotes the smallest extension closed subcategory of $\mathcal{D}$ that contains S.

- The $n \times n$ anti-diagonal matrix with entries a_k, $1 \leq k \leq n$ is defined by

$$\mathrm{Adiag}(a_1, \ldots, a_n)_{ij} = \begin{cases} a_k & \text{if } i = k, j = n + 1 - k \\ 0 & \text{otherwise.} \end{cases}.$$

That is, $\mathrm{Adiag}(a_1, \ldots, a_n) = \begin{pmatrix} 0 & \cdots & 0 & a_1 \\ 0 & \cdots & a_2 & 0 \\ \vdots & ⋰ & \vdots & 0 \\ a_n & \cdots & 0 & 0 \end{pmatrix}.$

- For $0 \leq i \leq \dim X$,

$$\mathrm{Coh}^{\leq i}(X) = \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) \leq i\},$$

$$\begin{aligned}
\mathrm{Coh}^{\geq i}(X) &= \{E \in \mathrm{Coh}(X) : \text{for } 0 \neq F \subset E,\ \dim \mathrm{Supp}(F) \geq i\}, \\
\mathrm{Coh}^{i}(X) &= \mathrm{Coh}^{\leq i}(X) \cap \mathrm{Coh}^{\geq i}(X).
\end{aligned}$$

- For an interval $I \subset \mathbb{R} \cup \{+\infty\}$,

$$\mathrm{HN}^{\mu}_{\omega,B}(I) = \{E \in \mathrm{Coh}(X) : [\mu^{-}_{\omega,B}(E), \mu^{+}_{\omega,B}(E)] \subset I\}.$$

  Similarly the subcategory $\mathrm{HN}^{\nu}_{\omega,B}(I) \subset \mathcal{B}_{\omega,B}$ is defined.

- For a Fourier-Mukai functor $\Upsilon : D^b(X) \to D^b(X)$ and the heart $\mathfrak{A}$ of a t-structure for which $D^b(X) \cong D^b(\mathfrak{A})$,

$$\Upsilon^k_{\mathfrak{A}}(E) = H^k_{\mathfrak{A}}(\Upsilon(E)).$$

  For a sequence of integers $i_1, \ldots, i_s$,

$$V^{\Upsilon}_{\mathfrak{A}}(i_1, \ldots, i_s) = \{E \in D^b(X) : \Upsilon^j_{\mathfrak{A}}(E) = 0 \text{ for } j \notin \{i_1, \ldots, i_s\}\}.$$

  If Υ is a Fourier-Mukai transform then $E \in \mathrm{Coh}(X)$ being Υ-WIT$_i$ is equivalent to $E \in V^{\Upsilon}_{\mathrm{Coh}(X)}(i)$.

- For a g-dimensional polarized projective variety (X, L) with Picard rank one, the Chern character of any $E \in D^b(X)$ is of the form $(a_0, a_1\ell, a_2\ell^2/2!, \ldots, a_g\ell^g/g!)$ for some $a_i \in \mathbb{Z}$. Here $\ell = c_1(L)$. For simplicity we write

$$\mathrm{ch}(E) = (a_0, a_1, a_2, \ldots, a_g).$$

- When F is a locally free sheaf, sometimes we abuse notation to write F for the functor $F \otimes (-)$.

Chapter 1

Introduction

1.1 Context

Sheaves and derived categories

Algebraic Geometry is the study of geometric objects called varieties which arise as zero sets of polynomial equations. There are many objects associated to a variety X which behave well with respect to restrictions to open subsets of X. Sheaves are a special class of objects among them, defined in terms of local information with global compatibilities. The most interesting class of sheaves are the coherent sheaves which form an abelian category usually denoted by $\mathrm{Coh}(X)$. Due to a classical result of Gabriel, this category is a strong invariant for X (see [Gab]). More specifically, one can reconstruct a smooth projective variety X from its coherent sheaves. As a result, it is sensible to study geometric questions about X through its coherent sheaves. This is more viable with the tools in homological algebra. In particular, the theory of derived categories provides an efficient algebraic platform to study the geometry of a variety indirectly. This machinery was introduced by Grothendieck's student Verdier in 1960s (see [Ver]). Throughout this thesis we denote the bounded derived category of coherent sheaves on a variety X by $D^b(X)$. The derived category $D^b(X)$ can be considered as a reasonable invariant of the variety X.

Some of the recent developments on derived categories were highly influenced by important ideas from Mathematical Physics. Moreover, these studies have been extremely successful in cross-fertilizing both Physics and Mathematics by creating deep interconnections. One of the earliest work in this area is Kontsevich's homological mirror symmetry conjecture (see [Kon]). The aim was to establish a mathematical framework which gives a conceptual understanding of mirror symmetry. His original proposal conjectures an equivalence of two categories, namely, the bounded derived category of a Calabi-Yau threefold and the derived Fukaya category of its mirror partner which is also a Calabi-Yau threefold. Furthermore, the complexes in derived categories arise as important objects in mathematical string theory. In particular, sheaf-theoretic models of D-branes can be interpreted as objects in derived categories (see [Asp2]). The study of D-branes on Calabi-Yau manifolds inspired various mathematical questions and one of them is to find a new categorical stability notion. Motivated by Douglas's work on Π-stability for D-branes (see [Dou]), Bridgeland introduced the notion of stability conditions on triangulated categories (see [Bri1]).

Categorical stability

One of the most important things in Algebraic Geometry is the study of moduli spaces of geometric objects. Usually these objects can be isomorphism classes of certain type of varieties, schemes, morphisms, vector bundles, sheaves or complexes of these kind of geometric objects. First rigorous constructions of moduli spaces were done in the 1960s mainly by Mumford. In particular, he introduced Geometric Invariant Theory (GIT) in order to construct moduli spaces (see [MFK]). Since then this subject has continued to develop rapidly and most importantly with influential ideas from physics after the 1980s.

One of the first examples is Mumford's GIT construction of moduli spaces of vector bundles. Here he introduced the notion of stability for vector bundles on a smooth projective curve (see [Mum1]). Later on, Takemoto, Gieseker, Maruyama and Simpson generalized this to coherent sheaves on higher dimensional projective varieties (see [Tak, Gie1, Mar, Sim2]). When E is vector bundle on a smooth projective curve C, one can consider two topological invariants of E, namely, the rank $\operatorname{rk}(E)$ and the degree $\deg(E)$. Mumford defined the slope $\mu(E)$ of E as the quotient $\deg(E)/\operatorname{rk}(E)$. A vector bundle E on C is said to be slope (semi)stable if $\mu(E') < (\leq)\, \mu(E)$ for all non-trivial proper subbundles E' of E. One of the most important properties of slope stability is that, any coherent sheaf fits into a unique filtration, called the Harder-Narasimhan filtration, with semistable quotients (see [HN]). Also any semistable sheaf fits into a Jordan-Hölder filtration with stable sheaves as quotient objects. Therefore, one can consider stable sheaves simply as the building blocks of all coherent sheaves. Mumford used the techniques in GIT to construct the coarse moduli space $M_C(r,d)$ of semistable bundles with rank r and degree d. The notion of slope stability for vector bundles on C can be extended to all coherent sheaves on C by setting $\mu = +\infty$ for all torsion sheaves on C.

The notion of Bridgeland stability can be interpreted essentially as an abstraction of the usual slope stability for sheaves. Briefly, a stability condition σ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying certain axioms. Here Z is called the central charge function and $\mathcal{P}$ is called the slicing of $\mathcal{D}$. Bridgeland showed that, equivalently, one can define a stability condition on $\mathcal{D}$ by giving a bounded t-structure on $\mathcal{D}$ and a stability function Z on its heart $\mathcal{A}$ satisfying the Harder-Narasimhan property. Then the stability condition σ is usually written as the pair $(Z, \mathcal{P})$ or $(Z, \mathcal{A})$. Bridgeland also showed that the space $\operatorname{Stab} \mathcal{D}$ of all stability conditions on $\mathcal{D}$ has a natural topology. Moreover, it is a complex manifold (possibly infinite dimensional). When C is a smooth projective curve, Bridgeland observed that usual slope stability on $\operatorname{Coh}(C)$ can be interpreted in his categorical notion by defining the stability function $Z(E) = -\deg(E) + i \operatorname{rk}(E)$ for $E \in \operatorname{Coh}(C)$. So the pair $(Z, \operatorname{Coh}(C))$ defines a stability condition on the derived category $D^b(C)$.

Some results about Bridgeland stability

There are many known examples of Bridgeland stability conditions on varieties. For a smooth projective curve with positive genus, Macrì showed that essentially there exists only one stability condition (see [Macr1]) and this result was previously shown for an elliptic curve by Bridgeland (see [Bri1]). This is exactly the induced stability condition on the derived category from the usual slope stability. More precisely, the stability manifold is isomorphic to the product $\mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ is the complex upper half plane. However, Okada proved that the stability manifold of the projective line is isomorphic to $\mathbb{C}^2$ (see [Oka]).

The category of coherent sheaves does not arise as a heart of a Bridgeland stability condition for higher dimensional smooth projective varieties. In particular, slope stability does not give rise to a stability condition on the derived category. So more work is needed to construct the hearts for stability conditions on varieties of dimension above one. In homological algebra, there is a way to obtain new hearts of t-structures from the known ones by the process called tilting with respect to a torsion pair (see [HRS]). Tilting has been extremely successful in constructing stability condition hearts. It was first used by Bridgeland in his construction of stability conditions on $K3$ and abelian surfaces (see [Bri2]). Later on, Arcara and Bertram extended those ideas to all smooth projective surfaces (see [AB]).

Construction of Bridgeland stability conditions on a given projective threefold is an important problem. However, unlike for a projective surface, there is no known construction which gives stability conditions for all projective threefolds. A conjectural construction of Bridgeland stability conditions for any projective threefold was introduced by Bayer, Macrì and Toda in [BMT], and the problem is reduced to proving an inequality, which the authors call a Bogomolov-Gieseker type inequality, holds for certain tilt stable objects. This inequality also appears to have many other interesting applications and a proof of Fujita's Conjecture for threefolds is one of them (see [BBMT]). It has been shown to hold for three dimensional projective space (see [BMT] and [Macr2]), and smooth quadric threefold (see [Sch]), some progress has been made for more general threefolds (see [Tod3, LM, Tod5, Tod6]).

The group of all derived autoequivalences of a variety contains lots of geometric information, and most of the time it is very complicated to study. Bridgeland stability turned out to be extremely useful to study this group. For example, Bridgeland gave an explicit conjectural description of this group for a $K3$ surface, and now it is known to be true when the Picard rank is one (see [BB]). There is another reason to study Bridgeland stability conditions, which is to address a classical question in the study of derived categories. When a smooth projective variety has an ample canonical or anticanonical bundle, Bondal and Orlov showed that the original variety can be reconstructed from its derived category (see [BO]). It is then natural to ask for other varieties, how much extra data one needs to reconstruct the variety from its derived category. This is very much close to the explicit understanding of "point like objects" which was introduced in [BO]. In this general setting, it is expected that the extra data for the reconstruction is given by fixing a stability condition on the derived category (see [Asp1]).

One of the crucial implications of having a Bridgeland stability condition is that it enables us to single out certain objects in the triangulated category which can possibly be parameterized as moduli spaces. Lots of studies have been done on moduli spaces of complexes of sheaves both with and without a notion of stability (see [HL, Secton

4.D.V] for further details). Most interestingly, the notion of Bridgeland stability is extremely useful to realize families of such moduli spaces. Also there is a well-behaved wall and chamber structure in the space of Bridgeland stability conditions. Bridgeland conjecturally proposed that the moduli space of semistable objects with a fixed numerical class is a coarse moduli space for any smooth projective variety (see [Bri2]). Lots of studies have been done in this direction, and for example, Bayer and Macrì completed the proof of projectivity of such moduli spaces for a $K3$ surface (see [BaM2]). Also they realized changes of stability conditions produce birational maps between those moduli spaces (see [BaM3]).

Another area which has a growing interest in Bridgeland stability is on defining new invariants to a variety through the moduli spaces of complexes of sheaves. The Donaldson-Thomas (DT) invariants were originally introduced as counting invariants of holomorphic vector bundles on a Calabi-Yau threefold. In the algebro-geometric definition, DT theory depends on the choice of stability condition on coherent sheaves and classically it is given by the choice of an ample divisor. Then, the wall crossing formula describes the behaviour of DT invariants under the change of stability conditions. Bridgeland stability turns out to produce a systematic study of this wall crossing phenomena by extending the classical setup to stability conditions on the derived categories. See [Tod4] for further details.

Fourier-Mukai theory

The notion of Fourier-Mukai transform (FM transform for short) was introduced by Mukai in early 1980s (see [Muk2]). He showed that the Poincaré bundle induces a non-trivial equivalence between the derived categories of an abelian variety and its dual variety. In general, when X and Y are smooth projective varieties, any object $\mathcal{E} \in D^b(X \times Y)$ induces a functor from $D^b(X)$ to $D^b(Y)$ defined by $\Phi_{\mathcal{E}}^{X \to Y}(-) = \mathbf{R}p_{2*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-))$. Here p_i, $i = 1, 2$ are the projection maps from $X \times Y$ to X and Y, respectively. Usually the functor $\Phi_{\mathcal{E}}^{X \to Y}$ is called a Fourier-Mukai functor (FM functor for short) and when it is an equivalence of the categories it is commonly called a Fourier-Mukai transform. One can think about the Fourier-Mukai transform as an algebro-geometric version of the Fourier transform in mathematical analysis.

One of the remarkable results in this field is Orlov's representability theorem, which says any derived equivalence of two smooth projective varieties is actually an FM transform (see [Orl1]). Most of the time, the group of derived autoequivalences can be an interesting object to study the geometry of a variety. However, when a projective variety has an ample canonical or anticanonical bundle, Bondal and Orlov showed that this group consists of only trivial equivalences (see [BO]). In other words, it is generated by the derived equivalences coming from the automorphisms of the variety, twists with line bundles and the shift functor of the derived category. But, in other cases, for example when a projective variety has trivial canonical bundle, the group of derived autoequivalences may not be that simple and can be very interesting to study. For abelian varieties, Orlov explicitly studied this group through the isometric automorphisms of the variety (see [Orl2]).

Mukai himself used his transform to study various geometric questions about abelian varieties (see [Muk2, Muk3, Muk4]). Since then Fourier-Mukai theory turned out to be extremely successful in studying stable sheaves and their moduli spaces. In [Bri1], Bridgeland realized that the stability manifold of a triangulated category carries a natural left action of the autoequivalences of the category. So Fourier-Mukai techniques have been very successful in studying moduli problems of complexes. In these studies, sometimes it is useful to understand the group of derived autoequivalences via some suitable representations. For example, FM transforms' induced transforms on the cohomological ring which are important in the study of the stability of complexes.

1.2 Outline

The aim of this section is to briefly discuss the important contents of this thesis.

Conjectural construction on threefolds

In [BMT], following the ideas in mathematical physics (see [AD, Section 2.3]) together with the known results for dimension two case (see [AB]), the authors conjectured that the function

$$Z_{\omega,B}(E) = -\int_X e^{-B-i\omega}\,\mathrm{ch}(E)$$

is a central charge function of some Bridgeland stability condition on any smooth projective variety X. Here $B + i\omega \in \mathrm{NS}_{\mathbb{C}}(X)$ is a complexified ample class. When X is a threefold, they conjecturally constructed a heart $\mathcal{A}_{\omega,B}$ for $Z_{\omega,B}$ as a double tilt of $\mathrm{Coh}(X)$, motivated by Bridgeland's construction for $K3$ and abelian surfaces.

The first tilt of $\mathrm{Coh}(X)$ associated to the Harder-Narasimhan filtration with respect to the twisted slope $\mu_{\omega,B}$ stability is denoted by $\mathcal{B}_{\omega,B}$. They proved that abelian category $\mathcal{B}_{\omega,B}$ of two term complexes is Noetherian. If the twisted Chern character with respect to B is defined by $\mathrm{ch}^B(-) = e^{-B}\,\mathrm{ch}(-)$, then they observed that the vector $(\omega^2\,\mathrm{ch}_1^B, \Im Z_{\omega,B}, -\Re Z_{\omega,B})$ for objects in $\mathcal{B}_{\omega,B}$ behaves like the Chern character vector $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ for coherent sheaves on a surface. Consequently, they introduced the notion of tilt slope $\nu_{\omega,B}$ stability for objects in $\mathcal{B}_{\omega,B}$, and showed that the Harder-Narasimhan property holds for tilt stability. The conjectural stability condition heart $\mathcal{A}_{\omega,B}$ is the tilt of $\mathcal{B}_{\omega,B}$ associated to the Harder-Narasimhan filtration with respect to the tilt slope $\nu_{\omega,B}$ stability. There is another way to construct the same abelian category $\mathcal{A}_{\omega,B}$ by using the notion of polynomial stability and see [BMT, Section 4] for details. This construction is important to show that $\mathcal{A}_{\omega,B}$ is Noetherian. By $\mathscr{C}_{\omega,B}$ we denote the class of tilt stable objects $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$. The pair $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$ defines a Bridgeland stability condition on X if and only if any $E \in \mathscr{C}_{\omega,B}$ satisfies the so-called *Weak Bogomolov-Gieseker Type Inequality*:

$$\Re Z_{\omega,B}(E[1]) < 0, \text{ that is } \mathrm{ch}_3^B(E) < \frac{\omega^2}{2}\,\mathrm{ch}_1^B(E).$$

In [BMT], the authors also proposed the following strong inequality for objects in $\mathscr{C}_{\omega,B}$.

Conjecture 1.1. (Bayer, Macrì and Toda, [BMT, Conjecture 1.3.1]) *Any $E \in \mathscr{C}_{\omega,B}$ satisfies the so-called Strong Bogomolov-Gieseker Type Inequality:*

$$\mathrm{ch}_3^B(E) \leq \frac{\omega^2}{18}\,\mathrm{ch}_1^B(E).$$

This strong inequality is already known to hold for projective 3-space (see [Macr2]) and the smooth quadric threefold (see [Sch]). The main goal of this thesis is to prove the strong Bogomolov-Gieseker type inequality for a principally polarized abelian threefold with Picard rank one.

Minimal objects and Bogomolov-Gieseker type inequalities

For a given smooth projective threefold X, $\mathscr{M}_{\omega,B}$ denotes the class of tilt slope $\nu_{\omega,B}$ stable objects $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$ and $\mathrm{Ext}^1_X(\mathcal{O}_x, E) = 0$ for all $x \in X$. In Chapter 3, we show that the objects in $\mathscr{M}_{\omega,B}[1]$ are minimal objects (also called simple objects in the literature) in $\mathcal{A}_{\omega,B}$ (see Lemma 3.5). Moreover, we show that for any object $E \in \mathscr{C}_{\omega,B}$ there exists $E' \in \mathscr{M}_{\omega,B}$ such that $0 \to E \to E' \to T \to 0$ is a short exact sequence in $\mathcal{B}_{\omega,B}$ for some $T \in \mathrm{Coh}^0(X)$ (see Proposition 3.13). Therefore, we only need to check the Bogomolov-Gieseker type inequalities for objects in $\mathscr{M}_{\omega,B}$.

This reduction result is analogous to the requirement of the usual Bogomolov-Gieseker inequality only for slope stable locally free sheaves on a surface S, because every torsion free sheaf E fits into a short exact sequence $0 \to E \to E^{**} \to T \to 0$ in $\mathrm{Coh}(S)$ for some $T \in \mathrm{Coh}^0(S)$. Furthermore, $E^{**}[1]$ is a minimal object of some stability condition heart of the surface, which is a tilt of coherent sheaves associated to the Harder-Narasimhan filtration with respect to the slope stability (see Example 3.2 (2) or [Huy2]).

Cohomological Fourier-Mukai transform

For an abelian variety X, the group $\mathrm{Aut}\, D^b(X)$ of FM transforms from X to itself is well understood via the notion of isometric isomorphism (see [Orl2] or [Huy1, Chapter 9]). To any FM transform $\Phi_{\mathcal{E}}$ in $\mathrm{Aut}\, D^b(X)$ with kernel $\mathcal{E}$, Orlov constructed an isometric automorphism $f_{\mathcal{E}}$ of the product $X \times \widehat{X}$. He showed that $\Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}$ is a surjective map of groups, and its kernel consists of certain trivial FM transforms (which sends skyscraper sheaves to skyscraper sheaves up to shift) generated by the shifts, translations and twists by line bundles in $\mathrm{Pic}^0(X)$. When (X, L) is a principally polarized g-dimensional abelian variety, let $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ be the central $\mathbb{Z}$-extension of the group $\mathrm{SL}(2,\mathbb{Z})$ generated by the FM transform Φ from X to X with kernel the Poincaré bundle on $X \times X$, $L \otimes (-)$ and $[1]$ as a subgroup of $\mathrm{Aut}\, D^b(X)$. So $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ is a subgroup of $\mathrm{Aut}\, D^b(X)$ and one can canonically identify the isometric automorphisms of it with elements from $\mathrm{SL}(2,\mathbb{Z})$. Any FM transform in $\mathrm{Aut}\, D^b(X)$ induces a linear isomorphism on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$, called the cohomological Fourier-Mukai transform, and this gives rise to a representation of $\mathrm{Aut}\, D^b(X)$.

When (X, L) is a g-dimensional principally polarized abelian variety with Picard rank one, we prove the following in Chapter 4:

Theorem 1.2 (=4.9)**.** *If we write the Chern character of any object in $D^b(X)$ as a vector with respect to the basis $\left\{\ell^k/k!\right\}_{0 \le k \le g}$, then there is a group homomorphism*

$$(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})} \to \mathrm{GL}(H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})) \quad \Phi_{\mathcal{E}} \mapsto \rho^{(g)}(f_{\mathcal{E}}),$$

such that the induced cohomological FM transform $\Phi^{\mathrm{H}}_{\mathcal{E}} = \pm \rho^{(g)}(f_{\mathcal{E}})$. Here $\rho^{(g)}$ is a *variant of $(g+1)$-dimensional symmetric power representation of* $\mathrm{GL}(2,\mathbb{R})$ *which has an explicit matrix description.*

Therefore, the cohomological FM transform $\Phi_{\mathcal{E}}^{\mathrm{H}}$ of an FM transform $\Phi_{\mathcal{E}}$ in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\operatorname{Aut} D^b(X)$ is $\rho^{(g)} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for some integers x, y, z, w with $xw - yz = 1$. When $\Phi_{\mathcal{E}}$ is a non-trivial FM transform or equivalently when $y \neq 0$, we get the following:

$$\operatorname{ch}^{-w\ell/y}\left(\Phi_{\mathcal{E}}(E)\right) = (-1)^g y^g \operatorname{Adiag}\left(1, \frac{-1}{y^2}, \ldots, \frac{(-1)^{g-1}}{y^{2(g-1)}}, \frac{(-1)^g}{y^{2g}}\right) \operatorname{ch}^{x\ell/y}(E). \tag{1.1}$$

This allows us to handle the numerology in the same way as that of the classical FM transform with kernel the Poincaré line bundle, which is crucial in this thesis. A matrix representation for the induced transform of an abelian surface was also considered in [YY].

FM transforms and Bridgeland stability on abelian threefolds

The space of all stability conditions of a variety X carries a natural left action of the group $\operatorname{Aut} D^b(X)$. This can be defined via a natural left action $(\Phi_{\mathcal{E}} \cdot W)([E]) = W\left([\Phi_{\mathcal{E}}^{-1}(E)]\right)$ of $\Phi_{\mathcal{E}} \in \operatorname{Aut} D^b(X)$ on $W \in \operatorname{Hom}(K(X), \mathbb{C})$. Recall that, when $\alpha \in \mathrm{NS}_{\mathbb{C}}(X)$ is a complexified ample class on a smooth projective variety X, it is expected that $Z_\alpha(-) = -\int_X e^{-\alpha} \operatorname{ch}(-)$ defines a central charge function of some Bridgeland stability condition on X. In Chapter 5, when (X, L) is a g-dimensional principally polarized abelian variety with Picard rank one, we view the action of the FM transform $\Phi_{\mathcal{E}}$ in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\operatorname{Aut} D^b(X)$, on Z_α explicitly as:

$$\Phi_{\mathcal{E}} \cdot Z_\alpha = \zeta Z_{\alpha'} \tag{1.2}$$

for some $\alpha' \in \mathrm{NS}_{\mathbb{C}}(X)$ and $\zeta \in \mathbb{C}^*$ (see Proposition 5.1). (A similar result for dimension 2 case is obtained in [MYY, Appendix].) When ζ is real one can expect that the FM transform $\Phi_{\mathcal{E}}$ gives an equivalence of some hearts of particular stability conditions of $D^b(X)$ whose α and α' are determined by $\Im \zeta = 0$. Following these ideas we conjecture the following in general.

Conjecture 1.3. *Let X be a g-dimensional abelian variety. For a given FM transform $\Phi_{\mathcal{E}}$ in $\operatorname{Aut} D^b(X)$, there exist $(g-1)$-fold tilts $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\operatorname{Coh}(X)$, and $q \in \mathbb{Z}$ such* *that*

$$\Phi_{\mathcal{E}}(\mathcal{A}_1) \cong \mathcal{A}_2[q].$$

In this thesis, we use the techniques of Fourier-Mukai theory to prove this conjecture for principally polarized abelian surfaces and threefolds with Picard rank one. The result for any abelian surface is already known due to Huybrechts (see [Huy2]), but we give an alternative proof in the spirit of the thesis. Minimal objects of the abelian subcategories are sent to minimal objects again under an FM transform. This enables us to obtain an inequality involving the top part of the Chern character of minimal objects in these abelian categories. When $g = 2$, by the Hirzebruch-Riemann-Roch formula, this is the usual Bogomolov-Gieseker inequality for slope stable vector bundles. For a principally polarized abelian threefold with Picard rank one, it is exactly the strong Bogomolov-Gieseker type inequality for tilt stable objects in $\mathscr{M}_{\omega,B}$. Therefore, we get the following.

Theorem 1.4 (= 5.7)**.** *Any tilt stable object with zero tilt slope satisfies the strong Bogomolov-Gieseker type inequality for a principally polarized abelian threefold with Picard rank one.*

Consequently, we obtain the main theorem of this thesis:

Theorem 1.5 (= 5.8)**.** *Let (X, L) be a principally polarized abelian threefold with Picard rank one and let ℓ be $c_1(L)$. Let α, β such that $\alpha/\sqrt{3} \in \mathbb{Q}_{>0}$ and $\beta \in \mathbb{Q}$. Let $\mathcal{A}_{\alpha\ell,\beta\ell}$ be the double tilt of* $\mathrm{Coh}(X)$ *as introduced by Bayer, Macrì and Toda. Then the pair $(\mathcal{A}_{\alpha\ell,\beta\ell}, Z_{\beta\ell+i\alpha\ell})$ defines a Bridgeland stability condition on $D^b(X)$.*

Equivalences of abelian categories

The main goal of this thesis is to prove Theorem 1.4, and for that we need to establish the equivalence of the abelian subcategories in Conjecture 1.3 for the abelian threefold. Most of the techniques that we use are influenced by the proof of the equivalences of the abelian categories for an abelian surface. The aim of Chapter 6 is to get some familiarization with them and we closely follow the proof of Yoshioka (see [Yos]).

When (X, L) is a g-dimensional principally polarized abelian variety with Picard rank one, we let $\mathcal{F}_0$ denote the subcategory of torsion-free sheaves on X all of whose Harder-Narasimhan semistable factors have slope less than or equal to 0, and $\mathcal{T}_0$ denotes the subcategory of sheaves whose torsion-free parts have Harder-Narasimhan semistable factors of slope greater than 0. If Φ is the FM transform from X to X with kernel the Poincaré line bundle on the product $X \times X$, then $\Phi \circ \Phi \cong (-1)^*[-g]$. So we have the convergence of the Mukai spectral sequence: $E_2^{p,q} = \Phi^p_{\mathrm{Coh}(X)} \Phi^q_{\mathrm{Coh}(X)}(E) \Longrightarrow H^{p+q-g}_{\mathrm{Coh}(X)}((-1)^*E)$ for E.

When $g = 2$, the tilt of the torsion pair $(\mathcal{T}_0, \mathcal{F}_0)$ is denoted by $\mathcal{A}_0 = \langle \mathcal{F}_0[1], \mathcal{T}_0 \rangle$. We use the Mukai spectral sequence to study the slope stability of sheaves in $\mathcal{F}_0$ and $\mathcal{T}_0$ under the FM transform Φ. In particular, we show that $\Phi[1](\mathcal{F}_0) \subset \mathcal{A}_0[-1]$ and $\Phi[1](\mathcal{T}_0) \subset \mathcal{A}_0$, and so the equivalence $\Phi[1](\mathcal{A}_0) \cong \mathcal{A}_0$. Then we use the anti-diagonal representation (1.1) of any non-trivial cohomological FM transform to show that the Conjecture 1.3 holds for any FM transform.

Understanding the homological Fourier-Mukai transform for the case of $g = 3$ is central to this thesis. In Chapter 7, we study the slope stability of sheaves in $\mathcal{F}_0$ and $\mathcal{T}_0$ under the FM transform Φ. In particular, we investigate the images under Φ of the torsion sheaves supported in dimension one and two. In Chapter 8, we extend those results to any non-trivial FM transform by using the anti-diagonal representation (1.1) of the induced cohomological FM transform. Let $\alpha = B + i\omega$, $\alpha' = B' + i\omega'$ be the solution sets of $\Im \zeta = 0$ in (1.2) for a non-trivial FM transform Γ. Let $\widehat{\Gamma}$ be the quasi inverse of FM transform $\Gamma[2]$. At the end of Chapter 8, we show that the images of the abelian category $\mathcal{B}_{\omega,B}$ (and $\mathcal{B}_{\omega',B'}$ respectively) under the FM transform Γ (and $\widehat{\Gamma}$ respectively) have non-zero cohomologies with respect to $\mathcal{B}_{\omega',B'}$ (and $\mathcal{B}_{\omega,B}$ respectively) only in positions 0, 1 and 2. Since we have the isomorphisms $\Gamma \circ \widehat{\Gamma} \cong [-2]$ and $\widehat{\Gamma} \circ \Gamma \cong [-2]$, the abelian categories $\mathcal{B}_{\omega,B}$ and $\mathcal{B}_{\omega',B'}$ behave somewhat similarly to the category of coherent sheaves on an abelian surface under the FM transforms. Finally, in Chapter 9, we prove that the FM transform Γ gives the equivalence $\Gamma[1](\mathcal{A}_{\omega,B}) \cong \mathcal{A}_{\omega',B'}$ of abelian categories.

1.3 Note

Most of the results in this thesis are also available in our papers [MP1, MP2].

Chapter 2

Preliminaries

The aim of this chapter is to introduce the essential preliminary notions which we shall need in the main context of this thesis.

2.1 Slope stability on sheaves

This section contains various results associated to the notion of slope stability for coherent sheaves. Slope stability is also known as Mumford-Takemoto stability and was introduced in [Mum1, Tak]. We adopt similar notation to [BMT].

Let X be a smooth projective variety of dimension n. Let $B \in \mathrm{NS}_{\mathbb{Q}}(X)$ and $\omega \in \mathrm{NS}_{\mathbb{R}}(X)$ be an ample class on X. The *twisted Chern character* ch^B with respect to B is defined by

$$\mathrm{ch}^B(-) = e^{-B}\,\mathrm{ch}(-).$$

Definition 2.1. The *twisted slope* $\mu_{\omega,B}$ on $\mathrm{Coh}(X)$ is defined by

$$\mu_{\omega,B}(E) = \begin{cases} +\infty & \text{if } E \text{ is a torsion sheaf} \\ \frac{\omega^{n-1}\,\mathrm{ch}_1^B(E)}{\mathrm{ch}_0^B(E)} & \text{otherwise} \end{cases}$$

for $E \in \mathrm{Coh}(X)$.

When $B = 0$ the twisted slope $\mu_{\omega,0}$ is usually called the *slope* with respect to ω on $\mathrm{Coh}(X)$. A coherent sheaf E is said to be $\mu_{\omega,B}$*-(semi)stable* (also called slope (semi)stable), if

$$\text{for any } 0 \neq F \subsetneq E \text{ in } \mathrm{Coh}(X), \text{ we have } \mu_{\omega,B}(F) < (\leq)\, \mu_{\omega,B}(E/F).$$

The following classical result is crucial in this thesis.

Theorem 2.2. (Usual Bogomolov-Gieseker Inequality) *Any $\mu_{\omega,B}$-semistable torsion free sheaf E satisfies the so-called usual Bogomolov-Gieseker inequality*

$$\omega^{n-2}\left(\operatorname{ch}_1^B(E)^2 - 2\operatorname{ch}_0^B(E)\operatorname{ch}_2^B(E)\right) \geq 0.$$

See [Rei, Bog, Gie2, Lan] or [HL, Section 3.4] for further details on this inequality.

Lemma 2.3. (Harder-Narasimhan Filtration) *Every coherent sheaf E fits into a unique finite chain of subobjects, called the Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that the factors $F_k = E_k/E_{k-1}$ are $\mu_{\omega,B}$-semistable sheaves with

$$\mu_{\omega,B}(F_1) > \mu_{\omega,B}(F_2) > \cdots > \mu_{\omega,B}(F_{m-1}) > \mu_{\omega,B}(F_m).$$

Lemma 2.4. (Jordan-Hölder Filtration) *Every semistable torsion free sheaf E fits into a finite chain of subobjects, called a Jordan-Hölder filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that the factors $F_k = E_k/E_{k-1}$ are $\mu_{\omega,B}$-stable sheaves with $\mu_{\omega,B}(F_k) = \mu_{\omega,B}(E)$ for all k.

Classically the notion of slope stability is defined only for torsion free sheaves. However, in [BMT], the authors introduced the above setting to extend it to all coherent sheaves. This can be verified by considering the torsion free quotient of any coherent sheaf by its maximal torsion subsheaf.

The Harder-Narasimhan filtration of $E \in \operatorname{Coh}(X)$ with respect to $\mu_{\omega,B}$-stability enables us to define the following slopes:

$$\left.\begin{aligned} \mu^+_{\omega,B}(E) &= \max_{0 \neq G \subseteq E} \ \mu_{\omega,B}(G) \\ \mu^-_{\omega,B}(E) &= \min_{G \subsetneq E} \ \mu_{\omega,B}(E/G) \end{aligned}\right\}.$$

Then one can show that for two coherent sheaves E, F on X,

$$\text{if } \mu^{-}_{\omega,B}(E) > \mu^{+}_{\omega,B}(F) \text{ then } \operatorname{Hom}_X(E,F) = 0.$$

Definition 2.5. For an interval $I \subset \mathbb{R} \cup \{+\infty\}$, the subcategory $\mathrm{HN}^{\mu}_{\omega,B}(I) \subset \mathrm{Coh}(X)$ is defined by

$$\mathrm{HN}^{\mu}_{\omega,B}(I) = \{E \in \mathrm{Coh}(X) : [\mu^{-}_{\omega,B}(E), \mu^{+}_{\omega,B}(E)] \subset I\},$$

and for $t \in \mathbb{R} \cup \{+\infty\}$, we write $\mathrm{HN}^{\mu}_{\omega,B}(t) = \mathrm{HN}^{\mu}_{\omega,B}([t,t])$. Define the subcategories $\mathcal{T}_{\omega,B}$ and $\mathcal{F}_{\omega,B}$ of $\mathrm{Coh}(X)$ by setting

$$\left.\begin{aligned} \mathcal{T}_{\omega,B} &= \mathrm{HN}^{\mu}_{\omega,B}((0,+\infty]) \\ \mathcal{F}_{\omega,B} &= \mathrm{HN}^{\mu}_{\omega,B}((-\infty,0]) \end{aligned}\right\}.$$

Note 2.6. There is another classical notion of stability for coherent sheaves on a smooth polarized projective variety (X, L), called *Gieseker stability.* A sheaf $E \in \mathrm{Coh}^{d}(X)$ is said to be Gieseker (semi)stable, if for any proper subsheaf $0 \neq F \subset E$, we have $p_F(m) < (\leq)\, p_E(m)$ for $m >> 0$. Here p_E is called the *reduced Hilbert polynomial* of E and it is the unique monic polynomial associated to the *Hilbert polynomial* P_E defined by $P_E(m) = \chi(E \otimes L^m)$. We have the following implications for torsion free sheaves:

$$\text{slope stable} \Longrightarrow \text{Gieseker stable} \Longrightarrow \text{Gieseker semistable} \Longrightarrow \text{slope semistable}.$$

See [HL, Section 1.2] for details.

2.2 Some homological algebra

A *triangulated category* $\mathcal{D}$ is an additive category equipped with a shift functor, and a class of triangles, called distinguished triangles satisfying certain axioms. We denote the shift functor by $[1] : \mathcal{D} \to \mathcal{D}$, and write a distinguished triangle as $A \to B \to C \to A[1]$. See [Mil, Huy1, GM, Har1, Ver] for details. The bounded derived categories of coherent sheaves on smooth projective varieties are the most important examples of triangulated categories in this thesis.

A *t-structure* on $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ such that, if we let $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$, for $n \in \mathbb{Z}$, then we have

(i) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$, $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$,

(ii) $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$,

(iii) for any $X \in \mathcal{D}$ there exists a distinguished triangle $A \to X \to B \to A[1]$ such that $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

The *heart* $\mathcal{H}$ of this t-structure is $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. The t-structure is called *bounded* if

$$\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \mathcal{D} = \bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\geq n}.$$

It is known that the heart $\mathcal{H}$ is an abelian category (see [GM, Theorem IV.4.4] or [Mil, Theorem 1.2.1]), and also a bounded t-structure is determined by its heart (see [Bri2, Lemma 3.1]). So we denote the i-th cohomology of $X \in \mathcal{D}$ with respect to the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ by $H^i_{\mathcal{H}}(X)$.

If $A \to B \to C \to A[1]$ is a distinguished triangle in $\mathcal{D}$, then we have the exact sequence

$$\cdots \to H^{-1}_{\mathcal{H}}(C) \to H^0_{\mathcal{H}}(A) \to H^0_{\mathcal{H}}(B) \to H^0_{\mathcal{H}}(C) \to H^1_{\mathcal{H}}(A) \to \cdots$$

of cohomologies from $\mathcal{H}$ (see [GM, Theorem IV.4.11] or [Mil, Theorem 1.2.19]).

Example 2.7. Let $D^b(\mathcal{A})$ be the bounded derived category of an abelian category $\mathcal{A}$. Then the pair of subcategories

$$\left.\begin{aligned} \mathcal{D}^{\leq 0} &= \{X \in D^b(\mathcal{A}) : H^i_{\mathcal{A}}(X) = 0 \text{ for } i > 0\} \\ \mathcal{D}^{\geq 0} &= \{X \in D^b(\mathcal{A}) : H^i_{\mathcal{A}}(X) = 0 \text{ for } i < 0\} \end{aligned}\right\}$$

define a bounded t-structure on $D^b(\mathcal{A})$ and the corresponding heart is $\mathcal{A}$. This is called the *standard t-structure* on $D^b(\mathcal{A})$.

Let us discuss about the torsion theory of an abelian category. It provides a useful method, called tilting, to construct interesting t-structures from the known ones. This was first introduced by Happel, Reiten and Smalø in [HRS].

Definition 2.8. A *torsion pair* on an abelian category $\mathcal{A}$ is a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ such that

(i) $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for every $T \in \mathcal{T}$, $F \in \mathcal{F}$, and

(ii) every $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \to T \to E \to F \to 0$$

in $\mathcal{A}$ for some $T \in \mathcal{T}$, $F \in \mathcal{F}$.

Lemma 2.9. ([HRS, Proposition 2.1]) *Let $\mathcal{A}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$ and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\mathcal{A}$. Then the full subcategory defined by*

$$\mathcal{B} = \{X \in \mathcal{D} : H^i_{\mathcal{A}}(X) = 0 \text{ for } i \neq -1, 0,\ H^{-1}_{\mathcal{A}}(X) \in \mathcal{F},\ H^0_{\mathcal{A}}(X) \in \mathcal{T}\}$$

is the heart of bounded t-structure given by the pair of subcategories

$$\left.\begin{array}{l} \mathcal{D}^{\leq 0} = \{X \in \mathcal{D} : H^i_{\mathcal{A}}(X) = 0 \text{ for } i > 0,\ H^0_{\mathcal{A}}(X) \in \mathcal{T}\} \\ \mathcal{D}^{\geq 0} = \{X \in \mathcal{D} : H^i_{\mathcal{A}}(X) = 0 \text{ for } i < -1,\ H^{-1}_{\mathcal{A}}(X) \in \mathcal{F}\} \end{array}\right\}.$$

The abelian subcategory $\mathcal{B} \subset \mathcal{D}$ is usually called the *tilt* of $\mathcal{A}$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$ and we also write $\mathcal{B} = \langle \mathcal{F}[1], \mathcal{T} \rangle$. The t-structures defined by the hearts $\mathcal{A}$ and $\mathcal{B}$ give two different views for the objects in the triangulated category $\mathcal{D}$.

Example 2.10. For a smooth projective variety X, let $B \in \mathrm{NS}_{\mathbb{Q}}(X)$ and $\omega \in \mathrm{NS}_{\mathbb{R}}(X)$ be an ample class. Let $\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B}$ be the subcategories of $\mathrm{Coh}(X)$ defined with respect to $\mu_{\omega,B}$-stability as in Definition 2.5. Then it is straightforward to check that $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ defines a torsion pair on $\mathrm{Coh}(X)$, and let

$$\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle \subset D^b(X)$$

be the corresponding tilt of $\mathrm{Coh}(X)$.

The *Grothendieck group* $K(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the quotient of the free abelian group generated by the classes $[A]$ of objects $A \in \mathcal{A}$ modulo the relations given by $[A] + [C] = [B]$ for every short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. Similarly, the Grothendieck group $K(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the free abelian group generated by the classes $[A]$ of $A \in \mathcal{D}$ with the relations $[A] + [C] = [B]$ for every distinguished triangles $A \to B \to C \to A[1]$ in $\mathcal{D}$. If $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ then one can easily check that

$$K(\mathcal{D}) = K(\mathcal{A}).$$

Moreover, when $\mathcal{A} = \mathrm{Coh}(X)$ for a variety X we write $K(X) = K(\mathrm{Coh}(X))$.

2.3 Stability conditions on triangulated categories

In this section we recall important definitions and results associated to Bridgeland stability conditions. A detailed exposition of this material can be found in [Bri1, Tod2, Huy3, Bri3] and [BBR, Appendix D].

Let $\mathcal{D}$ be a triangulated category.

Definition 2.11. A *slicing* $\mathcal{P}$ of $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying

(1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbb{R}$,

(2) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

(3) for every $0 \neq E \in \mathcal{D}$, there exist $\phi_k \in \mathbb{R}$, $k = 1, \ldots, n$ such that $\phi_k > \phi_{k+1}$ and a set of distinguished triangles

$$E_{i-1} \to E_i \to A_i \to E_{i-1}[1], \quad i = 1, \ldots, n$$

in $\mathcal{D}$ with $E_0 = 0$, $E_n = E$ and $A_i \in \mathcal{P}(\phi_i)$ for all i.

We define

$$\phi^+_{\mathcal{P}}(E) = \phi_1 \text{ and } \phi^-_{\mathcal{P}}(E) = \phi_n,$$

and for any interval $I \subset \mathbb{R}$

$$\mathcal{P}(I) = \langle E \in \mathcal{P}(\phi) : \phi \in I \rangle \, .$$

Now it is not hard to see that for any $\phi \in \mathbb{R}$, the pairs $(\mathcal{P}((-\infty, \phi+1]), \mathcal{P}((\phi, +\infty)))$ and $(\mathcal{P}((-\infty, \phi+1)), \mathcal{P}([\phi, +\infty)))$ define t-structures on $\mathcal{D}$ with the hearts $\mathcal{P}((\phi, \phi+1])$ and $\mathcal{P}([\phi, \phi + 1))$ respectively. Usually, the abelian subcategory $\mathcal{P}((0, 1]) \subset \mathcal{D}$ is called the *heart of slicing* $\mathcal{P}$.

Definition 2.12. A *stability condition* σ on $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$, called the *central charge function*, and a *slicing* $\mathcal{P}$ of $\mathcal{D}$ such that if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)e^{i\pi\phi}$ for some $m(E) \in \mathbb{R}_{>0}$. The stability condition σ is usually written as a pair $(Z, \mathcal{P})$.

A stability condition $\sigma = (Z, \mathcal{P})$ is called *locally finite* if there exists $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ the quasi-abelian category $\mathcal{P}((t-\varepsilon, t+\varepsilon)) \subset \mathcal{D}$ is of finite length. We say σ is a *discrete* stability condition if the image of $Z : K(\mathcal{D}) \to \mathbb{C}$ is a discrete subgroup of $\mathbb{C}$.

Proposition 2.13. ([Bri2, Lemma 4.4]) *If σ is a discrete stability condition then it is locally finite.*

Let $\mathcal{A}$ be an abelian category.

Definition 2.14. A group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ is called a *stability function*, if for all $0 \neq E \in \mathcal{A}$,

$$Z(E) \in \{re^{i\pi\phi} : r \in \mathbb{R}_{>0} \text{ and } 0 < \phi \leq 1\} \subset \mathbb{C}.$$

The *phase* of $0 \neq E \in \mathcal{A}$ is defined by

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1].$$

An object $0 \neq E \in \mathcal{A}$ is called *(semi)stable*,

$$\text{if } \phi(A) < (\leq)\, \phi(E/A) \text{ for any } 0 \neq A \subsetneq E \text{ in } \mathcal{A}.$$

A *Harder-Narasimhan filtration* of $0 \neq E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E, \tag{2.1}$$

where factors $F_k = E_k/E_{k-1}$, $k = 1, \ldots, n$, are semistable in $\mathcal{A}$ with

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_{n-1}) > \phi(F_n).$$

The stability function Z satisfies the *Harder-Narasimhan property* for $\mathcal{A}$, if such a filtration exists for any non-trivial object in $\mathcal{A}$.

Remark 2.15. When the Harder-Narasimhan property holds for $\mathcal{A}$ with respect to the stability function Z, one can show that the filtration (2.1) is unique for a given $E \in \mathcal{A}$.

On the one hand, when we have a Bridgeland stability condition σ defined by a pair $(Z, \mathcal{P})$, the central charge function Z defines a stability function on the heart of σ with the Harder-Narasimhan property. On the other hand, a stability function on the heart $\mathcal{A}$ of a bounded t-structure, equipped with the Harder-Narasimhan property, can be easily extended to a central charge function, and the notion of phase on $\mathcal{A}$ can be extended to define a slicing of a Bridgeland stability condition. Therefore, we have the following.

Proposition 2.16. ([Bri1, Proposition 5.3]) *A stability condition on a triangulated category $\mathcal{D}$ can be defined equivalently by giving a bounded t-structure on $\mathcal{D}$ and a stability function on its heart with the Harder-Narasimhan property. Such a stability condition is denoted by the pair $(\mathcal{A}, Z)$.*

As a result of this theorem, for a Bridgeland stability condition, the stability function and the central charge function are exactly the same.

Let $\Upsilon \in \operatorname{Aut}\mathcal{D}$ and let $W : K(\mathcal{D}) \to \mathbb{C}$ be a group homomorphism. Then

$$(\Upsilon \cdot W)([E]) = W\left([\Upsilon^{-1}(E)]\right)$$

defines a left action of the group $\operatorname{Aut}\mathcal{D}$ on $\operatorname{Hom}(K(\mathcal{D}), \mathbb{C})$. Moreover, this can be extended to the natural left action of $\operatorname{Aut}\mathcal{D}$ on the space of all stability conditions on $\mathcal{D}$ by defining $\Upsilon \cdot (Z, \mathcal{A}) = (\Upsilon \cdot Z, \Upsilon(\mathcal{A}))$.

Let $\widetilde{\mathrm{GL}^+(2,\mathbb{R})}$ be the universal covering of $\mathrm{GL}^+(2,\mathbb{R})$. So any element $\theta \in \widetilde{\mathrm{GL}^+(2,\mathbb{R})}$ can be written as a pair $\theta = (T, f)$, where $f : \mathbb{R} \to \mathbb{R}$ is an increasing function with $f(\phi + 1) = f(\phi) + 1$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving linear isomorphism such that the induced map on $S^1 = \mathbb{R}/\mathbb{Z} = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}$ is fixed. One can get a new stability condition $(Z', \mathcal{P}')$ by setting $Z' = T^{-1} \circ Z$, $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$. This defines a right action of $\theta \in \widetilde{\mathrm{GL}^+(2,\mathbb{R})}$ on the stability condition $(Z, \mathcal{P})$.

The following proposition is useful to construct Bridgeland stability conditions on the bounded derived categories of coherent sheaves on varieties.

Proposition 2.17. ([BaM1, Proposition B.2]) *Let $\mathcal{A}$ be a heart of a bounded t-structure on $\mathcal{D}$, and let $Z : K(\mathcal{A}) \to \mathbb{C}$ be a stability function on $\mathcal{A}$. Let $\mathcal{A}_1$ be the full subcategory of objects in $\mathcal{A}$ with the phase $\phi = 1$. If we have*

(1) for $\mathcal{A}$, the image of $\Im Z$ is a discrete subgroup of $\mathbb{R}$,

(2) for any $A \in \mathcal{A}$, any sequence of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{j-1} \subset E_j \subset \cdots \subset A$$

with $E_k \in \mathcal{A}_1$ for all k stabilizes,

then the Harder-Narasimhan property holds for $\mathcal{A}$ with respect to Z.

2.4 Bridgeland stability conditions on varieties

Let X be a smooth projective variety and let $D^b(X)$ be the bounded derived category of coherent sheaves on X. A stability condition σ on $D^b(X)$ is *numerical* if the central charge function Z factors through the Chern character map $\mathrm{ch} : K(X) \to H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$:

$$\begin{tikzcd}
K(X) \arrow[rr, "Z"] \arrow[d, "\mathrm{ch}"] & & \mathbb{C} . \\
H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}) \arrow[urr, dashed] & &
\end{tikzcd}$$

A numerical stability condition σ satisfies the *support property* if there is a positive constant C such that for any σ-stable object E, we have

$$|| \mathrm{ch}(E) || \leq C |Z(E)|.$$

Here $|| - ||$ is a fixed norm on the vector space $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$.

Remark 2.18. Let $\mathrm{Stab}(X)$ be the space of all locally finite numerical stability conditions on $D^b(X)$. Bridgeland showed that $\mathrm{Stab}(X)$ has a natural topology, and also there is a local homeomorphism from $\mathrm{Stab}(X)$ to a subspace of the complexified numerical Grothendieck group of X. A stability condition $\sigma \in \mathrm{Stab}(X)$ is called *full*, if the connected component of $\mathrm{Stab}(X)$ containing σ has the same dimension as that of the complexified numerical Grothendieck group (see [Bri2] for further details). The support property was introduced by Kontsevich and Soibelman in [KS], and a stability condition in $\mathrm{Stab}(X)$ is full if and only if it satisfies the support property (see [BaM1, Proposition B.4]).

The following definition is adapted from [BaM1, Definition 2.1].

Definition 2.19. A stability condition σ on $D^b(X)$ is called *geometric* if

(i) all skyscraper sheaves $\mathcal{O}_x$ of $x \in X$ are σ-stable of the same phase, and

(ii) σ satisfies the support property.

We need the following result to obtain some properties of certain type of stability conditions on the bounded derived category of a variety.

Lemma 2.20. ([BrM, Proposition 5.4]) *Let X be a smooth projective variety and let $E \in D^b(X)$ be a non-zero object. If we have $\mathrm{Hom}_{\mathrm{X}}(E, \mathcal{O}_x[i]) = 0$ unless $0 \leq i \leq s$ for some integer $s \geq 0$, then E is quasi isomorphic to a complex*

$$0 \to F_{-s} \to F_{-s+1} \to \cdots \to F_0 \to 0$$

of locally free sheaves F_i.

Proposition 2.21. *Let X be a smooth projective variety of dimension n. Let $\sigma = (Z, \mathcal{P})$ be a locally finite numerical stability condition on $D^b(X)$ such that all skyscraper sheaves $\mathcal{O}_x$ of $x \in X$ are σ-stable with phase one. Then we have the following:*

(1) if $E \in \mathcal{P}((0,1])$ then $H^i_{\mathrm{Coh}(X)}(E) = 0$ for $i \notin \{-n+1, -n+2, \ldots, 0\}$,

(2) if $E \in \mathcal{P}(1)$ is σ-stable then $E \cong \mathcal{O}_x$ for some $x \in X$, or E is a complex such that $H^i_{\mathrm{Coh}(X)}(E) = 0$ for $i \notin \{-n+1, -n+2, \ldots, -1\}$, and

(3) if $E \in \mathrm{Coh}(X)$ then $E \in \mathcal{P}((-n+1, 1])$.

Proof. The following proof is adapted from [Bri2, Lemma 10.1].

Assume $E \in \mathcal{P}((0,1))$. Then for any skyscraper sheaf $\mathcal{O}_x$ of $x \in X$ we have $\mathcal{O}_x[i] \in \mathcal{P}(1+i)$ and $E[i] \in \mathcal{P}((i, 1+i))$. Therefore, for all $i < 0$, $\mathrm{Hom}_{\mathrm{X}}(E, \mathcal{O}_x[i]) = 0$, and $\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_x, E[1+i]) \cong \mathrm{Hom}_{\mathrm{X}}(E, \mathcal{O}_x[n-1-i])^* = 0$. So by Lemma 2.20, E is quasi isomorphic to a complex of locally free sheaves of length n.

Similarly, if $E \in \mathcal{P}(1)$ is σ-stable and not isomorphic to a skyscraper sheaf, then one can show that E is quasi isomorphic to a complex of locally free sheaves of length n. Moreover, we have $\mathrm{Hom}_{\mathrm{X}}(E, \mathcal{O}_x) = 0$ for all $x \in X$. So $H^0_{\mathrm{Coh}(X)}(E) = 0$ and E is quasi isomorphic to a complex of locally free sheaves of length $n-1$.

Let $E \in \mathrm{Coh}(X)$. For any $A \in \mathcal{P}((1, +\infty))$, we have $\mathrm{Hom}_{\mathrm{X}}(A, \mathcal{O}_x[-i]) = 0$ for all $x \in X$ and $i \geq 0$. Therefore $H^i_{\mathrm{Coh}(X)}(A) = 0$ for $i \geq 0$, and so $\mathrm{Hom}_{\mathrm{X}}(A, E) = 0$. Similarly, one can show that for any $B \in \mathcal{P}((-\infty, -n+1])$, $H^i_{\mathrm{Coh}(X)}(B) = 0$ for $i \leq 0$. Hence $\mathrm{Hom}_{\mathrm{X}}(E, B) = 0$.

This completes the proof of the proposition. □

Note 2.22. It is straightforward to check that σ-stable objects in $\mathcal{P}(1)$ are minimal objects in the heart $\mathcal{P}((0, 1])$ (see Definition 3.1). From the above proof, if $E \in \mathcal{P}(1)$ is stable then we have the following:

(i) when $\dim X = 1$, $E \cong \mathcal{O}_x$ for some $x \in X$;

(ii) when $\dim X = 2$, $E \cong \mathcal{O}_x$ for some $x \in X$ or $E \cong F_{-1}[1]$ for some locally free sheaf F_{-1}; and

(iii) when $\dim X = 3$, $E \cong \mathcal{O}_x$ for some $x \in X$ or E is quasi isomorphic to a complex $0 \to F_{-2} \overset{\alpha}{\to} F_{-1} \to 0$ of locally free sheaves. Then $\mathrm{im}(\alpha) \hookrightarrow F_{-1}$ and so $\mathrm{im}(\alpha)$ is a torsion free sheaf. Therefore, $\ker(\alpha) = H^{-2}_{\mathrm{Coh}(X)}(E)$ is a reflexive sheaf (from Lemma 2.46).

Example 2.23. Let C be a smooth projective curve. Then

$$Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$$

defines a stability function on $\mathrm{Coh}(C)$. The Harder-Narasimhan property holds for $\mathrm{Coh}(C)$ with respect to Z and it follows from Lemma 2.3 or Proposition 2.17. Therefore, the pair $(\mathrm{Coh}(C), Z)$ defines a geometric stability condition on $D^b(C)$.

However for $\dim X \geq 2$ we have the following.

Lemma 2.24. ([Tod1, Lemma 2.7]) *For a smooth projective variety X with $\dim X \geq 2$, there is no numerical stability condition on $D^b(X)$ with $\mathrm{Coh}(X)$ as the heart of a stability condition.*

When $\dim X = 2$, one can construct hearts of geometric stability conditions by tilting $\mathrm{Coh}(X)$.

Example 2.25. Let S be a smooth projective surface. Let $\omega, B \in \mathrm{NS}_{\mathbb{Q}}(S)$ and ω be an ample class. Let the abelian category $\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle \subset D^b(S)$ be the tilt of $\mathrm{Coh}(S)$ with respect to the torsion pair $\mathcal{T}_{\omega,B} = \mathrm{HN}^{\mu}_{\omega,B}((0,+\infty])$, $\mathcal{F}_{\omega,B} = \mathrm{HN}^{\mu}_{\omega,B}((-\infty,0])$ (see Example 2.10). Consider the central charge function $Z_{\omega,B}(-) = -\int_X e^{-B-i\omega}\operatorname{ch}(-)$. Since $\operatorname{ch}^B(-) = e^{-B}\operatorname{ch}(-)$,

$$Z_{\omega,B}(E) = \left(-\operatorname{ch}_2^B(E) + \frac{\omega^2}{2}\operatorname{ch}_0^B(E)\right) + i\omega\cdot\operatorname{ch}_1^B(E).$$

We have $\Im Z_{\omega,B}(E) \geq 0$ for $E \in \mathcal{T}_{\omega,B}$, where the equality holds when $E \in \mathrm{Coh}^0(X)$. But $\Re Z_{\omega,B}(E) < 0$ for $E \in \mathrm{Coh}^0(X)$. Also $\Im Z_{\omega,B}(E) \leq 0$ for $E \in \mathcal{F}_{\omega,B}$. Here the equality holds when E is a $\mu_{\omega,B}$-semistable torsion free sheaf with $\mu_{\omega,B}(E) = 0$ and from the usual Bogomolov-Gieseker inequality we have $\operatorname{ch}_2^B(E) \leq 0$ (see Theorem 2.2); so $\Re Z_{\omega,B}(E[1]) < 0$. Therefore, $Z_{\omega,B}$ defines a slope function on $\mathcal{B}_{\omega,B}$. See [AB, Corollary 2.1] for further details.

One can use Proposition 2.17 to show that the heart $\mathcal{B}_{\omega,B}$ satisfies the Harder-Narasimhan property with respect to $Z_{\omega,B}$. See [Bri2, Proposition 7.1] for details. Since ω, B are chosen to be rational classes, the pair $(Z_{\omega,B}, \mathcal{B}_{\omega,B})$ defines a discrete numerical stability condition on $D^b(S)$. Furthermore, one can check that the support property holds for it and so we obtain a family of geometric stability conditions on $D^b(S)$.

The following result is expected for any dimensional smooth projective variety X.

Conjecture 2.26. ([BMT, Conjecture 2.1.2], [Pol]) *If $B + i\omega \in \mathrm{NS}_{\mathbb{C}}(X)$ is a complexified ample class on X, then the function defined by*

$$Z_{\omega,B}(-) = -\int_X e^{-B-i\omega}\operatorname{ch}(-)$$

is a central charge function of some stability condition in $\mathrm{Stab}(X)$.

2.5 Conjectural construction on threefolds

Following the observations in Proposition 2.21 together with the expected central charge function in Conjecture 2.26, Bayer, Macrì and Toda conjecturally constructed Bridgeland stability conditions on any smooth projective threefold in [BMT]. Let us recall some of the important details of this conjectural construction.

Let X be a smooth projective threefold. Let $B \in \mathrm{NS}_{\mathbb{Q}}(X)$ and $\omega \in \mathrm{NS}_{\mathbb{R}}(X)$ be an ample class with ω^2 rational. If we write the twisted Chern character by $\operatorname{ch}^B(-) =$

$e^{-B}\operatorname{ch}(-)$, then its parts are given by

$$\left.\begin{array}{ll} \operatorname{ch}_0^B = \operatorname{ch}_0 & \operatorname{ch}_1^B = \operatorname{ch}_1 - B\operatorname{ch}_0 \\ \operatorname{ch}_2^B = \operatorname{ch}_2 - B\operatorname{ch}_1 + \dfrac{B^2}{2}\operatorname{ch}_0 & \operatorname{ch}_3^B = \operatorname{ch}_3 - B\operatorname{ch}_2 + \dfrac{B^2}{2}\operatorname{ch}_1 - \dfrac{B^3}{6}\operatorname{ch}_0 \end{array}\right\}.$$

In Section 1.1, we introduce the notion of twisted slope $\mu_{\omega,B}$ stability on $\operatorname{Coh}(X)$. By considering the Harder-Narasimhan filtrations of coherent sheaves on X, we define the subcategories $\mathcal{T}_{\omega,B}$, $\mathcal{F}_{\omega,B}$ of $\operatorname{Coh}(X)$. They form a torsion pair and let the abelian category $\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B}\rangle \subset D^b(X)$ be the corresponding tilt of $\operatorname{Coh}(X)$ (see Example 2.10).

Consider the central charge function $Z_{\omega,B} : K(X) \to \mathbb{C}$ defined by $Z_{\omega,B}(E) = -\int_X e^{-B-i\omega}\operatorname{ch}(E)$ as in Conjecture 2.26. We have

$$Z_{\omega,B}(E) = \left(-\operatorname{ch}_3^B(E) + \frac{\omega^2}{2}\operatorname{ch}_1^B(E)\right) + i\left(\omega\operatorname{ch}_2^B(E) - \frac{\omega^3}{6}\operatorname{ch}_0^B(E)\right).$$

The following result is very important.

Lemma 2.27. ([BMT, Lemma 3.2.1]) *For any $0 \neq E \in \mathcal{B}_{\omega,B}$, one of the following conditions holds:*

(i) $\omega^2\operatorname{ch}_1^B(E) > 0$,

(ii) $\omega^2\operatorname{ch}_1^B(E) = 0$ *and* $\Im Z_{\omega,B}(E) > 0$,

(iii) $\omega^2\operatorname{ch}_1^B(E) = \Im Z_{\omega,B}(E) = 0$, $-\Re Z_{\omega,B}(E) > 0$ *and* $E \cong T$ *for some* $0 \neq T \in \operatorname{Coh}^0(X)$.

As a result of this Lemma, in [BMT], the authors went on to remark that the vector

$$(\omega^2\operatorname{ch}_1^B, \Im Z_{\omega,B}, -\Re Z_{\omega,B})$$

for objects in $\mathcal{B}_{\omega,B}$ behaves like the Chern character vector $\operatorname{ch} = (\operatorname{ch}_0, \operatorname{ch}_1, \operatorname{ch}_2)$ for coherent sheaves on a surface. Consequently they introduce the notion of tilt stability on $\mathcal{B}_{\omega,B}$.

Definition 2.28. The *tilt-slope* $\nu_{\omega,B}$ on $\mathcal{B}_{\omega,B}$ is defined by

$$\nu_{\omega,B}(E) = \begin{cases} +\infty & \text{if } \omega^2\operatorname{ch}_1^B(E) = 0 \\ \frac{\Im Z_{\omega,B}(E)}{\omega^2\operatorname{ch}_1^B(E)} & \text{otherwise} \end{cases}$$

for $E \in \mathcal{B}_{\omega,B}$.

Then $E \in \mathcal{B}_{\omega,B}$ is said to be $\nu_{\omega,B}$-*(semi)stable* (also called tilt (semi)stable), if

$$\text{for any } 0 \neq F \subsetneq E \text{ in } \mathcal{B}_{\omega,B}, \text{ we have } \nu_{\omega,B}(F) < (\leq)\, \nu_{\omega,B}(E/F).$$

Lemma 2.29. ([BMT, Lemma 3.2.4]) *The Harder-Narasimhan property holds for the abelian category $\mathcal{B}_{\omega,B}$ with respect to the tilt-slope $\nu_{\omega,B}$ stability.*

As a result of this lemma, we define the following slopes for $E \in \mathcal{B}_{\omega,B}$:

$$\left.\begin{aligned} \nu^+_{\omega,B}(E) &= \max_{0 \neq G \subseteq E} \nu_{\omega,B}(G) \\ \nu^-_{\omega,B}(E) &= \min_{G \subsetneq E} \nu_{\omega,B}(E/G) \end{aligned}\right\}.$$

Then for an interval $I \subset \mathbb{R} \cup \{+\infty\}$, the subcategory $\mathrm{HN}^{\nu}_{\omega,B}(I) \subset \mathcal{B}_{\omega,B}$ is defined by

$$\mathrm{HN}^{\nu}_{\omega,B}(I) = \{E \in \mathcal{B}_{\omega,B} : [\nu^-_{\omega,B}(E), \nu^+_{\omega,B}(E)] \subset I\}.$$

We define the subcategories $\mathcal{T}'_{\omega,B}$ and $\mathcal{F}'_{\omega,B}$ of $\mathcal{B}_{\omega,B}$ by setting

$$\left.\begin{aligned} \mathcal{T}'_{\omega,B} &= \mathrm{HN}^{\nu}_{\omega,B}((0,+\infty]) \\ \mathcal{F}'_{\omega,B} &= \mathrm{HN}^{\nu}_{\omega,B}((-\infty,0]) \end{aligned}\right\}.$$

Then $(\mathcal{T}'_{\omega,B}, \mathcal{F}'_{\omega,B})$ forms a torsion pair on $\mathcal{B}_{\omega,B}$. Let the abelian category

$$\mathcal{A}_{\omega,B} = \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \rangle \subset D^b(X)$$

be the corresponding tilt of $\mathcal{B}_{\omega,B}$.

Conjecture 2.30. ([BMT, Conjecture 3.2.6]) *The pair $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$ is a Bridgeland stability condition on $D^b(X)$.*

Definition 2.31. Let $\mathscr{C}_{\omega,B}$ be the class of $\nu_{\omega,B}$-stable objects in $\mathcal{B}_{\omega,B}$ with the tilt slope $\nu_{\omega,B} = 0$.

Then $E[1] \in \mathcal{A}_{\omega,B}$ for any $E \in \mathscr{C}_{\omega,B}$.

Conjecture 2.32. ([BMT, Conjecture 3.2.7]) *Any $E \in \mathscr{C}_{\omega,B}$ satisfies the so called*

Weak Bogomolov-Gieseker Type Inequality:

$$\Re Z_{\omega,B}(E[1]) < 0, \text{ that is } \operatorname{ch}_3^B(E) < \frac{\omega^2}{2}\operatorname{ch}_1^B(E).$$

Moreover, in [BMT] the authors proposed the following strong inequality for objects in $\mathscr{C}_{\omega,B}$.

Conjecture 2.33. ([BMT, Conjecture 1.3.1]) *Any $E \in \mathscr{C}_{\omega,B}$ satisfies the so-called* ***Strong Bogomolov-Gieseker Type Inequality:***

$$\operatorname{ch}_3^B(E) \leq \frac{\omega^2}{18}\operatorname{ch}_1^B(E).$$

Since we have already chosen $B \in \mathrm{NS}_{\mathbb{Q}}(X)$ and $\omega \in \mathrm{NS}_{\mathbb{R}}(X)$ is an ample class with ω^2 is rational, the abelian category $\mathcal{A}_{\omega,B}$ satisfies the following important property. It was originally proved when ω is a rational class. However, a similar proof can be used when we have a weaker condition, namely ω^2 is rational. For example, a different parametrization given by $\omega \mapsto \sqrt{3}\omega$ is considered in [Macr2].

Lemma 2.34. ([BMT, Proposition 5.2.2]) *The abelian category $\mathcal{A}_{\omega,B}$ is Noetherian.*

As a corollary we have the following.

Corollary 2.35. ([BMT, Corollary 5.2.4]) *The Conjectures 2.30 and 2.32 are equivalent.*

2.6 Fourier-Mukai theory

This section contains a brief introduction to Fourier-Mukai theory. Further details can be found in [Huy1, BBR, BL2].

Let X, Y be smooth projective varieties and let p_i, $i = 1, 2$ be the projection maps from $X \times Y$ to X and Y, respectively. The *Fourier-Mukai functor* (FM functor for short) $\Phi_{\mathcal{E}}^{X\to Y} : D^b(X) \to D^b(Y)$ with kernel $\mathcal{E} \in D^b(X \times Y)$ is defined by

$$\Phi_{\mathcal{E}}^{X\to Y}(-) = \mathbf{R}p_{2*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)).$$

The map $\mathsf{Swap} : Y \times X \to X \times Y$ is defined by $\mathsf{Swap}(y, x) = (x, y)$ for any $x \in X$ and $y \in Y$.

Lemma 2.36. ([Huy1, Proposition 5.9]) *Let*

$$\mathcal{E}_L = \mathsf{Swap}^* \left(\mathbf{R}\,\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X\times Y}) \overset{\mathbf{L}}{\otimes} p_2^*\omega_Y \right) [\dim Y], \quad \textit{and}$$
$$\mathcal{E}_R = \mathsf{Swap}^* \left(\mathbf{R}\,\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X\times Y}) \overset{\mathbf{L}}{\otimes} p_1^*\omega_X \right) [\dim X].$$

Then we have the adjunctions

$$\Phi_{\mathcal{E}_L}^{Y\to X} \dashv \Phi_{\mathcal{E}}^{X\to Y} \dashv \Phi_{\mathcal{E}_R}^{Y\to X}.$$

When $\Phi_{\mathcal{E}}^{X\to Y}$ is an equivalence of the derived categories, usually it is called a *Fourier-Mukai transform* (FM transform for short). On the other hand we have the following crucial result.

Lemma 2.37. (Orlov's Representability Theorem, [Orl1]) *Any equivalence between $D^b(X)$ and $D^b(Y)$ is isomorphic to an FM transform $\Phi_{\mathcal{E}}^{X\to Y}$ for some $\mathcal{E} \in D^b(X\times Y)$.*

Any FM functor $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$ induces a linear map $\Phi_{\mathcal{E}}^{\mathrm{H}} : H^{2*}_{\mathrm{alg}}(X,\mathbb{Q}) \to H^{2*}_{\mathrm{alg}}(Y,\mathbb{Q})$ (sometimes called the cohomological FM functor) and it is an isomorphism when $\Phi_{\mathcal{E}}$ is an FM transform. The induced transform fits into the following commutative diagram, due to the Grothendieck-Riemann-Roch theorem.

$$\begin{tikzcd}
D^b(X) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "{[-]}"'] & D^b(Y) \arrow[d, "{[-]}"] \\
K(X) \arrow[r, "\Phi_{\mathcal{E}}^K"] \arrow[d, "v_X(-)"'] & K(Y) \arrow[d, "v_Y(-)"] \\
H^{2*}_{\mathrm{alg}}(X,\mathbb{Q}) \arrow[r, "\Phi_{\mathcal{E}}^{\mathrm{H}}"] & H^{2*}_{\mathrm{alg}}(Y,\mathbb{Q})
\end{tikzcd}$$

Here $v_Z(-) = \mathrm{ch}(-)\sqrt{\mathrm{td}_Z}$ is the Mukai vector map, where $\mathrm{ch} : K(Z) \to H^{2*}_{\mathrm{alg}}(Z,\mathbb{Q})$ is the Chern character map and td_Z is the Todd class of Z.

Let $v \in H^{2*}_{\mathrm{alg}}(X,\mathbb{Q})$ be a Mukai vector. Then $v = \sum_{i=0}^{\dim X} v_i$ for $v_i \in H^{2i}_{\mathrm{alg}}(X,\mathbb{Q})$ and the Mukai dual of v is defined by $v^* = \sum_{i=0}^{\dim X} (-1)^i v_i$. A symmetric bilinear form

$\langle -, - \rangle$ called *Mukai pairing* is defined by the formula

$$\langle v, w \rangle = -\int_X v^* \cdot w \cdot e^{c_1(X)/2}.$$

The following result is due to Mukai, Căldăraru and Willerton (see [Muk3, CW]).

Lemma 2.38. ([Huy1, Proposition 5.44], [CW]) *The induced transform on cohomology of an FM transform $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$ is an isometry with respect to the Mukai pairing. In other words, for any $E, F \in D^b(X)$*

$$\langle v_X([E])\,,\, v_X([F]) \rangle_X = \langle v_Y([\Phi_{\mathcal{E}}(E)])\,,\, v_Y([\Phi_{\mathcal{E}}(F)]) \rangle_Y\,.$$

2.7 Abelian varieties

The following material is standard and see [Mum2, BL2] for further details.

Over any field, an *abelian variety* X is a complete group variety, that is X is an algebraic variety equipped with the maps

- $m : X \times X \to X,\ (x, y) \mapsto x + y$ (the group law), and
- $(-1) : X \to X,\ x \mapsto -x$ (the inverse map),

together with the identity element $e \in X$.

For $a \in X$, the morphism $t_a : X \to X$ is defined by

$$t_a = m(-, a) : x \mapsto x + a.$$

Over the field of complex numbers, an abelian variety is a complex torus with the structure of a projective algebraic variety.

The Theorem of the Square says

$$\phi_L : X \to \mathrm{Pic}(X),\ x \mapsto t_x^* L \otimes L^{-1}$$

is a group homomorphism. The subgroup $\mathrm{Pic}^0(X)$ of $\mathrm{Pic}(X)$ is defined by

$$\mathrm{Pic}^0(X) = \{L \in \mathrm{Pic}(X) : \phi_L(x) \cong \mathcal{O}_X \text{ for all } x \in X\}$$

and it fits into the short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$$

of groups. The group $\mathrm{Pic}^0(X)$ is naturally isomorphic to an abelian variety called the *dual abelian variety* of X, denoted by $\widehat{X}$. Therefore, for any line bundle L on X, the map ϕ_L is an isogeny from X to $\widehat{X}$.

The *Poincaré line bundle* $\mathscr{P}$ on the product $X \times \widehat{X}$ is the uniquely determined line bundle satisfying

(i) $\mathscr{P}_{X\times\{\widehat{x}\}} \in \mathrm{Pic}(X)$ is represented by $\widehat{x} \in \widehat{X}$, and

(ii) $\mathscr{P}_{\{e\}\times\widehat{X}} \cong \mathcal{O}_{\widehat{X}}$.

In [Muk2], Mukai proved that the FM functor $\Phi_{\mathscr{P}}^{X\to\widehat{X}} : D^b(X) \to D^b(\widehat{X})$ is an equivalence of the derived categories, that is an FM transform.

A *polarized abelian variety* is a pair (X, L) such that X is an abelian variety and L is an ample line bundle on X. Then there is an induced polarization $\widehat{L}$ on the dual abelian variety $\widehat{X}$, see [BL1] for further details.

The polarization L is called *principal* when $\deg \phi_L = 1$. It is known that $\deg \phi_L = \chi(L)^2$. Therefore, for a g-dimensional principally polarized abelian variety (X, L), we have

$$\chi(L) = \ell^g/g! = 1,$$

where $\ell = c_1(L)$, and also the map $\phi_L : X \to \widehat{X}$ is an isomorphism. Moreover, $(\mathrm{id} \times \phi_L)^*\mathscr{P} \cong m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ is usually called the Poincaré line bundle on the product $X \times X$. We abuse notation by denoting it by $\mathscr{P}$, because it is unambiguous from the context. Let $\Phi : D^b(X) \to D^b(X)$ be the FM functor with the Poincaré line bundle on $X \times X$ as the kernel. Then

$$\Phi \cong \phi_L^* \circ \Phi_{\mathscr{P}}^{X\to\widehat{X}}$$

and so Φ is an autoequivalence of the derived category $D^b(X)$.

Lemma 2.39. ([Muk2, Muk4], [Huy1, Proposition 9.30, Lemma 9.23]) *We have the following:*

(1) $\Phi \circ \Phi \cong (-1)^*[-g]$,

(2) $(L \circ \Phi)^3 \cong [-g]$, *and*

(3) if we assume the Picard rank of X *is one, and the Chern character of* $E \in D^b(X)$ *is* $\mathrm{ch}(E) = (a_0, a_1\ell, a_2\ell^2/2!, \ldots, a_g\ell^g/g!)$, *then*

$$\mathrm{ch}(\Phi(E)) = \Phi^{\mathrm{H}}(\mathrm{ch}(E)) = (a_g, -a_{g-1}\ell, a_{g-2}\ell^2/2!, \ldots, (-1)^g a_0\ell^g/g!).$$

Lemma 2.40. ([Orl2], [Huy1, Proposition 9.53]) *Let* $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$ *be an FM transform between the derived categories of two abelian varieties* X, Y. *Then up to a shift its kernel* $\mathcal{E} \in D^b(X \times Y)$ *is isomorphic to either*

(i) a locally free sheaf on $X \times Y$, *or*

(ii) when $X \cong Y$, *a locally free sheaf on* X.

A vector bundle E on an abelian variety X is called *homogeneous* if we have $t_x^* E \cong E$ for all $x \in X$. A vector bundle E on X is homogeneous if and only if E can be filtered by line bundles from $\mathrm{Pic}^0(X)$ (see [Muk1]).

We call a vector bundle E is *semi-homogeneous* if for every $x \in X$ there exists a flat line bundle $\mathscr{P}_{X \times \{\widehat{x}\}}$ on X such that $t_x^* E \cong E \otimes \mathscr{P}_{X \times \{\widehat{x}\}}$.

A vector bundle E is called *simple* if we have $\mathrm{End}_{\mathrm{X}}(E) \cong \mathbb{C}$.

Lemma 2.41. ([Muk1, Theorem 5.8]) *Let* E *be a simple vector bundle on an abelian variety* X. *Then the following conditions are equivalent:*

(1) $\dim H^1(X, \mathcal{E}nd(E)) = g$,

(2) E *is semi-homogeneous,*

(3) $\mathcal{E}nd(E)$ *is a homogeneous vector bundle.*

Example 2.42. A restriction of a non-trivial Fourier-Mukai kernel on the product of two abelian varieties to a point is a semi-homogeneous bundle (see Lemma 2.40).

Lemma 2.43. ([Muk1, Propositions 6.13, 6.16]) *Let* (X, L) *be a polarized abelian variety and let* E *be a semi-homogeneous bundle on* X. *Then* E *is Gieseker semistable with respect to* L, *and if* E *is simple then it is slope stable with respect to* $c_1(L)$.

2.8 Some sheaf theory

In this thesis, we shall encounter reflexive sheaves at several occasions. The aim of this section is to recall some of the key properties of them.

Let X be a smooth projective variety of dimension n.

Lemma 2.44. ([Har2], [GH, Section 4, Chapter 5]) *Any coherent sheaf E on X admits a locally free resolution of length n. In other words, E fits into an exact sequence:*

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to E \to 0$$

for some locally free sheaves F_i on X.

For a coherent sheaf E on X, its dual is $E^* = \mathcal{H}om(E, \mathcal{O}_X)$. There is a natural map from any $E \in \mathrm{Coh}(X)$ to its double dual E^{**}, $E \to E^{**}$. If this map is an isomorphism then E is called a *reflexive* sheaf. When E is a torsion free sheaf, E injects into its double dual.

Lemma 2.45. ([OSS, Lemma 1.1.2]) *For any coherent sheaf E on X we have*

$$\dim \mathrm{Supp}\left(\mathcal{E}xt^i(E, \mathcal{O}_X)\right) \leq (n-i), \text{ for all } i.$$

The *singularity set* $\mathrm{Sing}(E)$ of a coherent sheaf $E \in \mathrm{Coh}(X)$ is defined as the locus where E is not locally free, that is

$$\mathrm{Sing}(E) = \{x \in X : \mathrm{Ext}^1_{\mathrm{X}}(E, \mathcal{O}_x) \neq 0\}.$$

This coincides with

$$S_{n-1}(E) = \bigcup_{i=1}^{n} \mathrm{Supp}\left(\mathcal{E}xt^i(E, \mathcal{O}_X)\right).$$

See [OSS, Chapter 2] for further details.

We collect some of the useful results about reflexive sheaves as follows.

Lemma 2.46. *We have the following:*

(1) if E is a reflexive sheaf then $\dim \mathrm{Sing}(E) \leq n-3$*;*

(2) a coherent sheaf E is reflexive if and only if it fits into a short exact sequence

$$0 \to E \to F \to G \to 0$$

in $\mathrm{Coh}(X)$ *for a locally free sheaf* F *and a torsion free sheaf* G*;*

(3) any $E \in \mathrm{Coh}(X)$ *fits into an exact sequence*

$$0 \to T \to E \to E^{**} \to Q \to 0$$

in $\mathrm{Coh}(X)$*, where* T *is the maximal torsion subsheaf of* E *and* Q *is a torsion sheaf supported in a subscheme of at least codimension* 2*;*

(4) for any $E \in \mathrm{Coh}(X)$*, its dual* E^* *is a reflexive sheaf;*

(5) any rank one reflexive sheaf is locally free, that is a line bundle.

See Propositions 1.1, 1.3, 1.9 and Corollary 1.2 of [Har3] for proofs of (2), (1), (5) and (4). The claim in (3) is an easy exercise.

When $\dim X = 3$, one can easily prove the following result which is useful in this thesis to identify reflexive sheaves.

Lemma 2.47. *Let* E *be a coherent sheaf on a smooth projective threefold* X*. Then* E *is reflexive if and only if*

(i) $\mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_x, E) = 0$ *for all* $x \in X$*, and*

(ii) $\mathrm{Ext}^2_{\mathrm{X}}(\mathcal{O}_x, E) \neq 0$ *for finitely many* $x \in X$*, that is* $\dim \mathrm{Sing}(E) \leq 0$*.*

The following result of Simpson's is very important in this thesis.

Lemma 2.48. ([Sim1, Theorem 2]) *Let* (X, L) *be a smooth projective variety of dimension* $n \geq 3$ *and let* ℓ *be* $c_1(L)$*. Let* E *be a slope semistable reflexive sheaf on* X *with* $\ell^{n-1}\,\mathrm{ch}_1(E) = 0$ *and* $\ell^{n-2}\,\mathrm{ch}_2(E) = 0$*. Then all Jordan-Hölder slope stable factors of* E *are locally free sheaves which have vanishing Chern classes.*

Chapter 3

Threefold Heart $\mathcal{A}_{\omega,B}$

The aim of this chapter is to realize some classes of minimal objects in the threefold heart $\mathcal{A}_{\omega,B}$ and then we reduce the requirement of Bogomolov-Gieseker type inequalities to certain type of tilt stable objects which are essentially some of these minimal objects. In the last section we obtain certain slope bounds for coherent cohomologies of the two step complexes in $\mathcal{B}_{\omega,B}$ when the Picard rank of X is one.

3.1 Some minimal objects of $\mathcal{A}_{\omega,B}$

First we recall the definition of a minimal object in an arbitrary abelian category.

Definition 3.1. Let $\mathcal{C}$ be an abelian category. Then a non-trivial object $A \in \mathcal{C}$ is said to be a *minimal object* if $0 \to E \to A \to F \to 0$ is a short exact sequence in $\mathcal{C}$ then $E = 0$ or $F = 0$. Equivalently, A is minimal when A has no proper subobjects.

Example 3.2. (1) For a smooth projective variety Y, all the skyscraper sheaves $\mathcal{O}_y$ on Y are the only minimal objects in $\mathrm{Coh}(Y)$.

(2) Let S be a smooth projective surface and let $\omega, B \in \mathrm{NS}_{\mathbb{Q}}(S)$ with ω be an ample class. The abelian subcategory $\mathcal{B}_{\omega,B}$ as defined in Example 2.10 has the following minimal objects (see [Huy2, Proposition 2.2]):

- skyscraper sheaves $\mathcal{O}_s$ of all $s \in S$, and
- $E[1]$, where E is a slope stable locally free sheaf with $\mu_{\omega,B}(E) = 0$.

Let X be a smooth projective threefold and let $B \in \mathrm{NS}_{\mathbb{Q}}(X)$ and $\omega \in \mathrm{NS}_{\mathbb{R}}(X)$ be an ample class with ω^2 is rational. Let the abelian category $\mathcal{A}_{\omega,B} \subset D^b(X)$ be the double tilt of $\mathrm{Coh}(X)$ as in the conjectural construction of Bridgeland stability conditions in Section 2.5.

Proposition 3.3. *For any $x \in X$, the skyscraper sheaf $\mathcal{O}_x$ is a minimal object in $\mathcal{A}_{\omega,B}$.*

Proof. For any $x \in X$, $\mathcal{O}_x \in \mathcal{T}_{\omega,B}$ and also $\mathcal{O}_x \in \mathcal{T}'_{\omega,B}$. Therefore, $\mathcal{O}_x \in \mathcal{A}_{\omega,B}$. Let

$$0 \to A \to \mathcal{O}_x \to B \to 0$$

be a short exact sequence in $\mathcal{A}_{\omega,B}$ such that $A \neq 0$. Now we need to show that $B = 0$.

We obtain the following long exact sequence of $\mathcal{B}_{\omega,B}$-cohomologies associated to the above short exact sequence in $\mathcal{A}_{\omega,B}$:

$$0 \to A_{-1} \to 0 \to B_{-1} \to A_0 \to \mathcal{O}_x \to B_0 \to 0.$$

Here $A_k = H^k_{\mathcal{B}_{\omega,B}}(A)$ and $B_k = H^k_{\mathcal{B}_{\omega,B}}(B)$. We have $A_{-1} = 0$ and so $A \cong A_0 \neq 0$. Let $C = A_0/B_{-1}$. Then

$$0 \to C \to \mathcal{O}_x \to B_0 \to 0$$

is a short exact sequence in $\mathcal{B}_{\omega,B}$. We obtain the following long exact sequence of $\mathrm{Coh}(X)$-cohomologies associated to the above short exact sequence in $\mathcal{B}_{\omega,B}$:

$$0 \to C^{-1} \to 0 \to B_0^{-1} \to C^0 \to \mathcal{O}_x \to B_0^0 \to 0.$$

Here $C^k = H^k_{\mathrm{Coh}(X)}(C)$ and $B_0^k = H^k_{\mathrm{Coh}(X)}(B_0)$. We have $C^{-1} = 0$ and so $C \cong C^0$.

If $B_0^0 \neq 0$ then $\mathcal{O}_x \cong B_0^0$ and $B_0^{-1} \cong C^0 \in \mathcal{T}_{\omega,B} \cap \mathcal{F}_{\omega,B} = \{0\}$. So $C = 0$ and $B_{-1} \cong A_0 \in \mathcal{T}'_{\omega,B} \cap \mathcal{F}'_{\omega,B} = \{0\}$ which implies $A_0 = 0$. This is not possible and so $B_0^0 = 0$. Therefore, $B_0 \cong B_0^{-1}[1]$, and

$$0 \to B_0^{-1} \to C^0 \to \mathcal{O}_x \to 0$$

is a short exact sequence in $\mathrm{Coh}(X)$. Here $\mathrm{ch}(\mathcal{O}_x) = (0,0,0,1)$. If $B_0^{-1} \neq 0$ then

$$0 \geq \mu_{\omega,B}(B_0^{-1}) = \mu_{\omega,B}(C^0) > 0.$$

This is not possible and so $B_0^{-1} = 0$ and $C^0 \cong \mathcal{O}_x$. Therefore, $B \cong B_{-1}[1]$, and we have the following short exact sequence in $\mathcal{B}_{\omega,B}$:

$$0 \to B_{-1} \to A_0 \to \mathcal{O}_x \to 0.$$

Since $\mathrm{ch}(\mathcal{O}_x) = (0,0,0,1)$, if $B_{-1} \neq 0$ then

$$0 \geq \nu_{\omega,B}(B_{-1}) = \nu_{\omega,B}(A_0) > 0.$$

This is not possible and so $B_{-1} = 0$. Therefore, $B = 0$, and so $\mathcal{O}_x \in \mathcal{A}_{\omega,B}$ is a minimal object as required. $\square$

We now identify further minimal objects.

Definition 3.4. Let $\mathscr{M}_{\omega,B}$ be the class of all objects $E \in \mathcal{B}_{\omega,B}$ such that

(i) E is $\nu_{\omega,B}$-stable,

(ii) $\nu_{\omega,B}(E) = 0$, and

(iii) $\mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_x, E) = 0$ for any skyscraper sheaf $\mathcal{O}_x$ of $x \in X$.

Recall Definition 2.31 that $\mathscr{C}_{\omega,B}$ is the class of tilt stable objects in $\mathcal{B}_{\omega,B}$ with $\nu_{\omega,B} = 0$. So clearly, $\mathscr{M}_{\omega,B} \subset \mathscr{C}_{\omega,B}$.

Lemma 3.5. *Let $E \in \mathscr{M}_{\omega,B}$. Then $E[1]$ is a minimal object of $\mathcal{A}_{\omega,B}$.*

Proof. By definition $\mathscr{M}_{\omega,B} \subset \mathcal{F}'_{\omega,B}$ and so $E[1] \in \mathcal{A}_{\omega,B}$. Let

$$0 \to A \to E[1] \to B \to 0$$

be a short exact sequence in $\mathcal{A}_{\omega,B}$ such that $B \neq 0$. Now we need to show that $A = 0$. We have the following long exact sequence of $\mathcal{B}_{\omega,B}$-cohomologies associated to the above short exact sequence in $\mathcal{A}_{\omega,B}$:

$$0 \to A_{-1} \to E \to B_{-1} \to A_0 \to 0 \to B_0 \to 0.$$

Here $A_k = H^k_{\mathcal{B}_{\omega,B}}(A)$ and $B_k = H^k_{\mathcal{B}_{\omega,B}}(B)$. We have $B_0 = 0$ and so $B \cong B_{-1}[1]$ which implies $B_{-1} \neq 0$.

Assume $A_{-1} \neq 0$. If $E/A_{-1} = 0$ then $B_{-1} \cong A_0 \in \mathcal{F}'_{\omega,B} \cap \mathcal{T}'_{\omega,B} = \{0\}$; but this is not possible as $B_{-1} \neq 0$. So $E/A_{-1} \neq 0$ and we have $E/A_{-1} \hookrightarrow B_{-1}$. Since $\nu^+_{\omega,B}(B_{-1}) \leq 0$,

$\nu_{\omega,B}(E/A_{-1}) \leq 0$. But this is not possible, because $A_{-1} \neq 0$ and E is $\nu_{\omega,B}$-stable with $\nu_{\omega,B}(E) = 0$. Therefore, $A_{-1} = 0$ and we have the following short exact sequence in $\mathcal{B}_{\omega,B}$:

$$0 \to E \to B_{-1} \to A_0 \to 0. \tag{3.1}$$

Assume $A_0 \neq 0$. Here $\nu_{\omega,B}(E) = 0$ implies $\omega^2 \operatorname{ch}_1^B(E) > 0$ and $\Im Z_{\omega,B}(E) = 0$. Then

$$0 \geq \nu_{\omega,B}(B_{-1}) = \frac{\Im Z_{\omega,B}(A_0)}{\omega^2 \operatorname{ch}_1^B(E) + \omega^2 \operatorname{ch}_1^B(A_0)}$$

implies $\Im Z_{\omega,B}(A_0) \leq 0$. If $\omega^2 \operatorname{ch}_1^B(A_0) \neq 0$ then $\nu_{\omega,B}(A_0) > 0$ implies $\Im Z_{\omega,B}(A_0) > 0$; which is not possible. Hence, $\omega^2 \operatorname{ch}_1^B(A_0) = 0$, and by Lemma 2.27, $\Im Z_{\omega,B}(A_0) \geq 0$. So $\Im Z_{\omega,B}(A_0) = 0$ and $A_0 \cong T$ for some $0 \neq T \in \operatorname{Coh}^0(X)$. Then short exact sequence (3.1) in $\mathcal{B}_{\omega,B}$ corresponds to an element from $\operatorname{Ext}^1_X(A_0, E) \cong \operatorname{Ext}^1_X(T, E)$. But we have $\operatorname{Ext}^1_X(\mathcal{O}_x, E) = 0$ for any $x \in X$ and so $\operatorname{Ext}^1_X(T, E) = 0$. Therefore, $B_{-1} \cong T \oplus E$, and so T is a subobject of B_{-1}. But this is not possible as $\nu_{\omega,B}(T) = +\infty$ and $E \in \mathscr{M}_{\omega,B}$. Therefore, $A_0 = 0$ and so $A = 0$ as required to complete the proof. □

Some classes of tilt stable candidates have been identified in [BMT] via the following definition.

Definition 3.6. For $E \in D^b(X)$ the discriminant $\overline{\Delta}_\omega$ in the sense of Drézet is defined by

$$\overline{\Delta}_\omega(E) = \left(\omega^2 \operatorname{ch}_1^B(E)\right)^2 - 2\omega^3 \operatorname{ch}_0^B(E) \cdot \omega \operatorname{ch}_2^B(E).$$

Proposition 3.7. ([BMT, Proposition 7.4.1]) *Let E be a $\mu_{\omega,B}$-stable locally free sheaf on X with $\overline{\Delta}_\omega(E) = 0$. Then either E or $E[1]$ in $\mathcal{B}_{\omega,B}$ is $\nu_{\omega,B}$-stable.*

Example 3.8. Let (X, L) be a polarized projective threefold, and let ℓ be $c_1(L)$. Then $\overline{\Delta}_{\frac{\sqrt{3}}{2}\ell}(\mathcal{O}_X) = \overline{\Delta}_{\frac{\sqrt{3}}{2}\ell}(L) = 0$. So by Proposition 3.7, $\mathcal{O}_X[1], L \in \mathcal{B}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ are $\nu_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$-stable. Also $\Im Z_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}(\mathcal{O}_X[1]) = \Im Z_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}(L) = 0$. Therefore, $\nu_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}(\mathcal{O}_X[1]) = \nu_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}(L) = 0$. So by Lemma 3.5, $\mathcal{O}_X[2], L[1] \in \mathcal{A}_{\frac{\sqrt{3}}{2}\ell, \frac{1}{2}\ell}$ are minimal objects.

Example 3.9. Let (X, L) be a principally polarized abelian threefold with Picard rank 1, and let ℓ be $c_1(L)$. Let $p, q \in \mathbb{Q}$ and $q > 0$. There exist simple semi-homogeneous vector bundles $\mathcal{E}_s^\pm$ parameterized by $s \in X$ having the Chern character $(u^3, u^2 v\ell, uv^2\ell^2/2, v^3\ell^3/6)$ with $u, v \in \mathbb{Z}$ such that $u > 0$, $\gcd(u, v) = 1$ and $v/u = p \pm q$. Here the vector bundles $\mathcal{E}_s^\pm$ are restrictions of the universal bundles $\mathcal{E}^\pm$ on $X \times X$ associated to some FM transforms. Also $\mathcal{E}_s^\pm$ are slope stable (see

Lemma 2.43). The discriminant in the sense of Drezet $\overline{\Delta}_{\sqrt{3}q\ell}(\mathcal{E}_s^{\pm}) = 0$, and so by Proposition 3.7, $\mathcal{E}_s^{+}, \mathcal{E}_s^{-}[1] \in \mathcal{B}_{\sqrt{3}q\ell,p\ell}$ are $\nu_{\sqrt{3}q\ell,p\ell}$-stable. Also we have $\Im Z_{\sqrt{3}q\ell,p\ell}(\mathcal{E}_s^{\pm}) = 0$ and $\mathrm{ch}_1^{p\ell}(\mathcal{E}_s^{\pm}) \neq 0$, and so $\nu_{\sqrt{3}q\ell,p\ell}(\mathcal{E}_s^{+}) = \nu_{\sqrt{3}q\ell,p\ell}(\mathcal{E}_s^{-}[1]) = 0$. Therefore, by Lemma 3.5, $\mathcal{E}_s^{+}[1], \mathcal{E}_s^{-}[2] \in \mathcal{A}_{\sqrt{3}q\ell,p\ell}$ are minimal objects.

Note 3.10. The tilt stable objects associated to minimal objects in Examples 3.8 and 3.9 clearly satisfy the corresponding Bogomolov-Gieseker type inequalities.

3.2 Reduction of the inequalities to minimal objects

The aim of this section is to reduce the requirement of Bogomolov-Gieseker type inequalities from $\mathscr{C}_{\omega,B}$ to its smaller subclass $\mathscr{M}_{\omega,B}$. The following propositions are important for us.

Proposition 3.11. ([LM, Proposition 3.1]) *Let $E \in \mathcal{B}_{\omega,B}$ be a $\nu_{\omega,B}$-semistable object with $\nu_{\omega,B}(E) < +\infty$. Then $H^{-1}_{\mathrm{Coh}(X)}(E)$ is a reflexive sheaf.*

Therefore, if $E \in \mathcal{A}_{\omega,B}$ is a minimal object not isomorphic to a skyscraper sheaf, then $H^{-2}_{\mathrm{Coh}(X)}(E)$ is a reflexive sheaf. Also see (iii) of Note 2.22.

Proposition 3.12. ([LM, Proposition 3.5]) *Let $0 \to E \to E' \to Q \to 0$ be a non splitting short exact sequence in $\mathcal{B}_{\omega,B}$ with $Q \in \mathrm{Coh}^0(X)$, $\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_x, E') = 0$ for any $x \in X$, and $\omega^2 \mathrm{ch}_1^B(E) \neq 0$. If E is $\nu_{\omega,B}$-stable then E' is $\nu_{\omega,B}$-stable.*

Recall that $\mathscr{C}_{\omega,B}$ is the class of $\nu_{\omega,B}$-stable objects $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$.

Proposition 3.13. *Let $E \in \mathscr{C}_{\omega,B}$. Then there exists $E' \in \mathscr{M}_{\omega,B}$ (that is $E'[1]$ is a minimal object in $\mathcal{A}_{\omega,B}$) such that*

$$0 \to E \to E' \to Q \to 0$$

is a short exact sequence in $\mathcal{B}_{\omega,B}$ for some $Q \in \mathrm{Coh}^0(X)$.

Proof. Let $E \in \mathscr{C}_{\omega,B} \setminus \mathscr{M}_{\omega,B}$. Assume the opposite of the claim in the proposition for E. Then there exists a sequence of non-splitting short exact sequences in $\mathcal{B}_{\omega,B}$, for $i \geq 1$

$$0 \to E_{i-1} \to E_i \to \mathcal{O}_{y_i} \to 0,$$

where $E_0 = E$, $E_i \in \mathscr{C}_{\omega,B}$ (see Proposition 3.12). So for each $i \geq 1$,

$$0 \to \mathcal{O}_{y_i} \to E_{i-1}[1] \to E_i[1] \to 0$$

is a short exact sequence in $\mathcal{A}_{\omega,B}$. Therefore,

$$E[1] = E_0[1] \twoheadrightarrow E_1[1] \twoheadrightarrow E_2[1] \twoheadrightarrow \cdots$$

is an infinite chain of quotients in $\mathcal{A}_{\omega,B}$. But this is not possible as $\mathcal{A}_{\omega,B}$ is Noetherian (see Lemma 2.34). This is the required contradiction. $\square$

Note 3.14. For any $Q \in \operatorname{Coh}^0(X)$ we have $\Re Z_{\omega,B}(Q) < 0$. So it follows that $E \in \mathscr{C}_{\omega,B}$ satisfies the (weak or strong) Bogomolov-Gieseker type inequality if the corresponding $E' \in \mathscr{M}_{\omega,B}$ in Proposition 3.13 satisfies the corresponding inequality.

3.3 Further properties of $\mathcal{A}_{\omega,B}$

Let (X, L) be a polarized projective threefold with Picard rank one and let ℓ be $c_1(L)$. Let D, B be in $\operatorname{NS}_{\mathbb{Q}}(X)$. Then there exists $b \in \mathbb{Q}$ such that $B = b\ell$. Assume $b > 0$. Then with respect to the twisted Chern character ch^D the central charge function is

$$Z_{\sqrt{3}B,D+B}(-) = -\int_X e^{-(B+i\sqrt{3}B)} \operatorname{ch}^D(-).$$

So for $E \in D^b(X)$, $\Im Z_{\sqrt{3}B,D+B}(E) = \sqrt{3}b\ell \left(\operatorname{ch}_2^D(E) - b\ell \operatorname{ch}_1^D(E)\right)$.

Proposition 3.15. *Let $E \in \mathcal{B}_{\sqrt{3}B,D+B}$ and let $E_i = H^i_{\operatorname{Coh}(X)}(E)$. Let $E_i^{\pm}$ be the Harder-Narasimhan semistable factors of E_i with highest and lowest $\mu_{\sqrt{3}B,D+B}$ slopes. Then we have the following:*

(1) if $E \in \operatorname{HN}^{\nu}_{\sqrt{3}B,D+B}((-\infty, 0))$, then $\ell^2 \operatorname{ch}_1^D(E_{-1}^+) < 0$;

(2) if $E \in \operatorname{HN}^{\nu}_{\sqrt{3}B,D+B}((0, +\infty])$ and $\operatorname{rk}(E_0) \neq 0$, then $\ell^2 \operatorname{ch}_1^D(E_0^-) > 2b\ell^3 \operatorname{ch}_0^D(E_0^-)$; and

(3) if E is tilt-stable with $\nu_{\sqrt{3}B,D+B}(E) = 0$, then

(i) $\ell^2 \operatorname{ch}_1^D(E_{-1}) \leq 0$ with equality if and only if $\operatorname{ch}_2^D(E_{-1}) = 0$, and

(ii) for $\operatorname{rk}(E_0) \neq 0$, $\ell^2 \operatorname{ch}_1^D(E_0) \geq 2b\ell^3 \operatorname{ch}_0^D(E_0)$ with equality if and only if $\operatorname{ch}_2^D(E_0) = 2b^2\ell^2 \operatorname{ch}_0^D(E_0)$.

Proof. $E \in \mathcal{B}_{\sqrt{3}B,D+B}$ fits in to the short exact sequence

$$0 \to E_{-1}[1] \to E \to E_0 \to 0$$

in $\mathcal{B}_{\sqrt{3}B,D+B}$.

(1) If $E \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((-\infty,0))$ then we have $E_{-1}[1] \in \mathrm{HN}^{\nu}_{\sqrt{3}B,B}((-\infty,0))$. From the Harder-Narasimhan filtration of E_{-1}, $E_{-1}[1]$ fits into the short exact sequence

$$0 \to E^{+}_{-1}[1] \to E_{-1}[1] \to Q[1] \to 0$$

in $\mathcal{B}_{\sqrt{3}B,D+B}$, where $Q = E_{-1}/E^{+}_{-1} \in \mathrm{Coh}(X)$.

Hence, $E^{+}_{-1}[1] \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((-\infty,0))$.

Let $\mathrm{ch}^{D}(E^{+}_{-1}) = (a_0, a_1\ell, a_2\ell^2/2, a_3\ell^3/6)$. Assume the opposite for a contradiction; so that $a_1 \geq 0$. We have

$$\begin{aligned}\nu_{\sqrt{3}B,D+B}(E^{+}_{-1}[1]) &= \frac{-\Im Z_{\sqrt{3}B,D+B}(E^{+}_{-1})}{-3B^2\,\mathrm{ch}_1^{D+B}(E^{+}_{-1})} \\ &= \frac{\sqrt{3}ba_1(ba_0 - a_1) + \frac{\sqrt{3}}{2}ba_1^2 + \frac{\sqrt{3}}{2}b(a_1^2 - a_0a_2)}{3a_0b^2(ba_0 - a_1)}.\end{aligned}$$

Since E^{+}_{-1} is $\mu_{\sqrt{3}B,D+B}$-semistable we have, by the usual Bogomolov-Gieseker inequality,

$$a_1^2 - a_0a_2 \geq 0,$$

and since $E^{+}_{-1} \in \mathcal{F}_{\sqrt{3}B,D+B}$ and $\nu_{\sqrt{3}B,D+B}(E^{+}_{-1}[1]) \neq +\infty$ we have $ba_0 - a_1 > 0$. Hence, as $a_0 > 0$, we have $\nu_{\sqrt{3}B,D+B}(E^{+}_{-1}[1]) \geq 0$.

But this is not possible as $E^{+}_{-1}[1] \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((-\infty,0))$. This is the required contradiction to complete the proof.

(2) Since $E \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((0,+\infty])$, $E_0 \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((0,+\infty])$. We have $0 \neq E_0^{-}$ is a torsion free quotient of E_0 as $\mathrm{rk}(E_0) > 0$. So

$$0 \to K \to E_0 \to E_0^{-} \to 0$$

is a short exact sequence in $\mathcal{B}_{\sqrt{3}B,D+B}$, where $K = \ker(E_0 \twoheadrightarrow E_0^{-}) \in \mathrm{Coh}(X)$. Since $E_0 \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((0,+\infty])$ we have $E_0^{-} \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((0,+\infty])$.

Let $\mathrm{ch}^D(E_0^-) = (a_0, a_1\ell, a_2\ell^2/2, a_3\ell^3/6)$. Assume the opposite for a contradiction; so that $a_1 \leq 2ba_0$. We have

$$\begin{aligned}\nu_{\sqrt{3}B,D+B}(E_0^-) &= \frac{\Im Z_{\sqrt{3}B,D+B}(E_0^-)}{3B^2\,\mathrm{ch}_1^{D+B}(E_0^-)} \\ &= \frac{-\frac{\sqrt{3}}{2}b(a_1^2 - a_0a_2) + \frac{\sqrt{3}}{2}ba_1(a_1 - 2ba_0)}{3b^2a_0(a_1 - ba_0)}.\end{aligned}$$

Here $E_0^- \in \mathcal{T}_{\sqrt{3}B,D+B}$ is torsion free which implies

$$a_1 - ba_0 > 0;$$

E_0^- is $\mu_{\sqrt{3}B,D+B}$-semistable which implies (by the usual Bogomolov-Gieseker inequality)

$$a_1^2 - a_0a_2 \geq 0.$$

Therefore, $\nu_{\sqrt{3}B,D+B}(E_0^-) \leq 0$.

But this is not possible as $E_0^- \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((0,+\infty])$. This is the required contradiction to complete the proof.

(3) Similar to (1) one can show that if $E \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}((-\infty,0])$ and $E_{-1} \neq 0$, then $\ell^2\,\mathrm{ch}_1^D(E_{-1}^+) \leq 0$. Therefore, for $E \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D+B}(0)$ we have $\ell^2\,\mathrm{ch}_1^D(E_{-1}) \leq 0$. The equality holds when E_{-1} is slope semistable, and so it satisfies the usual Bogomolov-Gieseker inequality. Since $\nu_{\sqrt{3}B,D+B}(E_{-1}) \leq 0$ we have $\ell^2\,\mathrm{ch}_1^D(E_{-1}) = 0$ if and only if $\mathrm{ch}_2^D(E_{-1}) = 0$.

Proof of (ii) is similar to that of (i).

□

We have

$$Z_{\sqrt{3}B,D-B}(-) = -\int_X e^{-(B+i\sqrt{3}B)}\,\mathrm{ch}^{D-2B}(-)$$

and also $\mathrm{ch}^{D-2B} = e^{2B}\,\mathrm{ch}^D$. Therefore, from the above proposition we get the following form of an equivalent proposition.

Proposition 3.16. *Let $E \in \mathcal{B}_{\sqrt{3}B,D-B}$ and let $E_i = H^i_{\mathrm{Coh}(X)}(E)$. Let $E_i^{\pm}$ be the Harder-Narasimhan semistable factors of E_i with highest and lowest $\mu_{\sqrt{3}B,D-B}$ slopes. Then we have the following:*

(1) if $E \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D-B}((-\infty,0))$*, then* $\ell^2 \operatorname{ch}_1^D(E_{-1}^+) < -2b\ell^3 \operatorname{ch}_0^D(E_{-1}^+)$*;*

(2) if $E \in \mathrm{HN}^{\nu}_{\sqrt{3}B,D-B}((0,+\infty])$ *and* $\operatorname{rk}(E_0) \neq 0$*, then* $\ell^2 \operatorname{ch}_1^D(E_0^-) > 0$*; and*

(3) if E *is tilt-stable with* $\nu_{\sqrt{3}B,D-B}(E) = 0$*, then*

  (i) $\ell^2 \operatorname{ch}_1^D(E_{-1}) \leq -2b\ell^3 \operatorname{ch}_0^D(E_{-1})$ *with equality if and only if* $\operatorname{ch}_2^D(E_{-1}) = 2b^2\ell^2 \operatorname{ch}_0^D(E_{-1})$*, and*

  (ii) for $\operatorname{rk}(E_0) \neq 0$*,* $\ell^2 \operatorname{ch}_1^D(E_0) \geq 0$ *with equality if and only if* $\operatorname{ch}_2^D(E_0) = 0$*.*

Chapter 4

Cohomological FM Transforms on Abelian Varieties

In this chapter, we first recall the Orlov's realization of all FM transforms on abelian varieties. Then we obtain an explicit matrix description for the cohomological FM transform on a principally polarized abelian variety with Picard rank one.

4.1 Group of FM transforms

Following the work of Orlov, the group of FM transforms on an abelian variety can be described explicitly as follows. We closely follow the details in [Orl2] and [Huy1, Chapter 9].

Let X, Y be two abelian varieties. Then one can write any morphism $f : X \times \widehat{X} \to Y \times \widehat{Y}$ as a matrix

$$f = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

with some morphisms $p : X \to Y$, $q : \widehat{X} \to Y$, $r : X \to \widehat{Y}$ and $s : \widehat{X} \to \widehat{Y}$. The morphism $\widetilde{f} : Y \times \widehat{Y} \to X \times \widehat{X}$ of f is defined by

$$\widetilde{f} = \begin{pmatrix} \widehat{s} & -\widehat{q} \\ -\widehat{r} & \widehat{p} \end{pmatrix}.$$

Then f is said to be *isometric* if it is an isomorphism and its inverse $f^{-1} \cong \widetilde{f}$. When $Y = X$, we denote the group of all isometric automorphisms of $X \times \widehat{X}$ by $U(X \times \widehat{X})$.

Let $\Phi_{\mathcal{E}}^{X \to Y}$ be an FM transform between two abelian varieties X and Y with kernel

$\mathcal{E} \in D^b(X \times Y)$. Let us define the map $\mu_X : X \times X \to X \times X$ by $\mu_X(x_1, x_2) = (x_1, m(x_1, x_2))$. Let $P_X = p_{14}^* \mathcal{O}_\Delta \otimes p_{23}^* \mathscr{P}_X$, where p_{ij} are the projection maps from $(X \times \widehat{X} \times X \times X)$, $\mathcal{O}_\Delta$ is the structure sheaf on the diagonal $\Delta \subset X \times X$, and $\mathscr{P}_X$ is the Poincaré bundle on $\widehat{X} \times X$. One can check that P_X induces an FM transform from $(X \times \widehat{X})$ to $(X \times X)$.

Let $\mathcal{F} \in D^b(X \times Y)$ be an object such that $\Phi_{\mathcal{F}}^{Y \to X} \cong \left(\Phi_{\mathcal{E}}^{X \to Y}\right)^{-1}$ and let $\mathrm{Ad}_{\mathcal{E}}$ be the FM transform from $X \times X$ to $Y \times Y$ with kernel $\mathcal{F} \boxtimes \mathcal{E}$. Then it satisfies

$$\Phi_{\mathrm{Ad}_{\mathcal{E}}(\mathcal{G})}^{Y \to Y} \cong \Phi_{\mathcal{E}}^{X \to Y} \circ \Phi_{\mathcal{G}}^{X \to X} \circ \left(\Phi_{\mathcal{E}}^{X \to Y}\right)^{-1}$$

for any $\mathcal{G} \in D^b(X \times X)$ (see [Orl2]). Now define the equivalence $F_{\mathcal{E}} : D^b(X \times \widehat{X}) \to D^b(Y \times \widehat{Y})$ by

$$F_{\mathcal{E}} = \left(\Phi_{P_Y}^{(Y \times \widehat{Y}) \to (Y \times Y)}\right)^{-1} \circ (\mathbf{R}\mu_{Y*})^{-1} \circ \mathrm{Ad}_{\mathcal{E}} \circ \mathbf{R}\mu_{X*} \circ \Phi_{P_X}^{(X \times \widehat{X}) \to (X \times X)},$$

so that $F_{\mathcal{E}}$ fits into the following commutative diagram (see [Huy1, Orl2]).

$$\begin{tikzcd}
D^b(X \times \widehat{X}) \arrow[r, "F_{\mathcal{E}}"] \arrow[d, "\Phi_{P_X}^{(X \times \widehat{X}) \to (X \times X)}"'] & D^b(Y \times \widehat{Y}) \arrow[d, "\Phi_{P_Y}^{(Y \times \widehat{Y}) \to (Y \times Y)}"] \\
D^b(X \times X) \arrow[d, "\mathbf{R}\mu_{X*}"'] & D^b(Y \times Y) \arrow[d, "\mathbf{R}\mu_{Y*}"] \\
D^b(X \times X) \arrow[r, "\mathrm{Ad}_{\mathcal{E}}"] & D^b(Y \times Y)
\end{tikzcd}$$

Orlov showed that the equivalence $F_{\mathcal{E}}$ can be expressed in a simple form as follows.

Lemma 4.1. ([Huy1, Proposition 9.39], [Orl2]) *The equivalence $F_{\mathcal{E}}$ is isomorphic to $f_{\mathcal{E}*}(-) \otimes N_{\mathcal{E}}$ for some line bundle $N_{\mathcal{E}}$ on $Y \times \widehat{Y}$ and isometric isomorphism $f_{\mathcal{E}} : X \times \widehat{X} \to Y \times \widehat{Y}$. Moreover, $f_{\mathcal{E}}(a, \widehat{a}) = (b, \widehat{b})$ if and only if $\Phi_{(b,\widehat{b})} \circ \Phi_{\mathcal{E}}^{X \to Y} \cong \Phi_{\mathcal{E}}^{X \to Y} \circ \Phi_{(a,\widehat{a})}$. Here $\Phi_{(z,\widehat{z})} = t_{z*}(-) \otimes \mathscr{P}_{\widehat{z}}$ and $\mathscr{P}_{\widehat{z}}$ is the restriction of the Poincaré line bundle on the product $Z \times \widehat{Z}$.*

Example 4.2. Let (X, L) be a principally polarized abelian variety. The following examples are important in this thesis (see [Huy1, Examples 9.38]). Here $\delta : X \to X \times X$ is the diagonal embedding.

$\Phi_{\mathcal{E}}$	$f_{\mathcal{E}}$	$N_{\mathcal{E}}$
$[1] = \Phi^{X\to X}_{\mathcal{O}_\Delta[1]}$	$f_{[1]} = \mathrm{id}_{X\times\widehat{X}}$	$\mathcal{O}_{X\times\widehat{X}}$
$\Phi_{(s,\widehat{s})} = t_{s*}(-) \otimes \mathscr{P}_{\widehat{s}}$	$f_{(s,\widehat{s})} = \mathrm{id}_{X\times\widehat{X}}$	$\mathscr{P}_{\widehat{s}} \boxtimes \mathscr{P}^*_s$
$\Phi = \Phi^{X\to X}_{\mathscr{P}}$	$f_{\mathscr{P}} = \begin{pmatrix} 0 & -\phi_L^{-1} \\ \phi_L & 0 \end{pmatrix}$	$\mathscr{P}_X$
$L \otimes (-) = \Phi^{X\to X}_{\delta_* L}$	$f_{\delta_* L} = \begin{pmatrix} 1 & 0 \\ -\phi_L & 1 \end{pmatrix}$	$L \boxtimes \mathcal{O}_{\widehat{X}}$

Let X, Y, Z be abelian varieties and let $\Phi^{X\to Y}_{\mathcal{E}}$, $\Phi^{Y\to Z}_{\mathcal{F}}$, $\Phi^{X\to Z}_{\mathcal{G}}$ be some FM transforms such that $\Phi^{X\to Z}_{\mathcal{G}} \cong \Phi^{Y\to Z}_{\mathcal{F}} \circ \Phi^{X\to Y}_{\mathcal{E}}$. Then one can show that $f_{\mathcal{G}} \cong f_{\mathcal{F}} \circ f_{\mathcal{E}}$ and $N_{\mathcal{G}} \cong N_{\mathcal{F}} \otimes f_{\mathcal{F}*} N_{\mathcal{E}}$ (see [Huy1, Exercise 9.41]). So there is a well defined group homomorphism

$$\sigma_X : \mathrm{Aut}\, D^b(X) \to U(X \times \widehat{X}),\ \Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}. \tag{4.1}$$

Lemma 4.3. ([Huy1, Proposition 9.55]) *The map σ_X is an epimorphism and its kernel consists of FM transforms $\Phi_{(s,\widehat{s})}[k]$ where $s \in X$, $\widehat{s} \in \widehat{X}$ and $k \in \mathbb{Z}$. So $\ker(\sigma_X) \cong \mathbb{Z} \oplus (X \times \widehat{X})$.*

Let (X, L) be a principally polarized g-dimensional abelian variety. Let $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ be the central $\mathbb{Z}$-extension of the group $\mathrm{SL}(2,\mathbb{Z})$ generated by Φ, $L \otimes (-)$ and $[1]$ as a subgroup of $\mathrm{Aut}\, D^b(X)$. So $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ is a subgroup of $\mathrm{Aut}\, D^b(X)$. The isometric automorphism of any FM transform in $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ is of the form

$$\begin{pmatrix} x & y\phi_L^{-1} \\ z\phi_L & w \end{pmatrix}$$

for some $x, y, z, w \in \mathbb{Z}$ satisfying $xw - yz = 1$.

Notation 4.4. In the rest of the thesis we abuse notation by dropping ϕ_L, ϕ_L^{-1} from this matrix.

So we have the following diagram (see [Huy1, Chapter 9]):

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \oplus (X \times \widehat{X}) \arrow[r] \arrow[d, equal] & \operatorname{Aut} D^b(X) \arrow[r] & U(X \times \widehat{X}) \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z} \oplus (X \times \widehat{X}) \arrow[r] & (X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})} \arrow[r] \arrow[u, hook] & \mathrm{SL}(2,\mathbb{Z}) \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

When $\operatorname{End}(X) \cong \mathbb{Z}$, it is known that $U(X \times \widehat{X}) \cong \mathrm{SL}(2,\mathbb{Z})$, and so $\operatorname{Aut} D^b(X) \cong (X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$. See [Orl2, Example 4.16] for further details.

4.2 Matrix representations of $\mathrm{GL}(2,\mathbb{R})$

Following [Kna], we explicitly construct a variant of the symmetric power representation of all dimensions (≥ 2) of $\mathrm{GL}(2,\mathbb{R})$.

For $k \geq 2$, let V_k be the vector space of homogeneous polynomials over $\mathbb{R}$ in variables u_1, u_2 of degree k. Then $V_k = \bigoplus_{r=0}^{k} \mathbb{R}\left(u_1^{k-r}u_2^r\right)$. So the set

$$\Omega = \left\{u_1^k, -\binom{k}{1}u_1^{k-1}u_2, \ldots, (-1)^r\binom{k}{r}u_1^{k-r}u_2^r, \ldots, (-1)^k u_2^k\right\}$$

is a basis of V_k. Here $\binom{k}{r} = \begin{cases} \frac{k!}{r!(k-r)!} & \text{if } 0 \leq r \leq k, \\ 0 & \text{otherwise.} \end{cases}$

We have $\dim_{\mathbb{R}} V_k = k+1$. Let us define the map

$$\rho^{(k)} : \mathrm{GL}(2,\mathbb{R}) \to \mathrm{GL}(V_k) \tag{4.2}$$

by

$$\rho^{(k)}(X)\left(Q\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = Q\left(X^{\mathrm{T}}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right),$$

for $X \in \mathrm{GL}(2,\mathbb{R})$ and $Q\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in V_k$. Then one can easily check that $\rho^{(k)}$ is a $(k+1)$-dimensional linear representation of $\mathrm{GL}(2,\mathbb{R})$. We now explicitly compute the matrix representation of $\rho^{(k)}$ with respect to the basis Ω. Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ and let $a_{m,n}^{(k)}$ be the (m,n)-entry of $\rho^{(k)}(X)$. By definition

$$(-1)^{n-1}\binom{k}{n-1}(xu_1+zu_2)^{k-n+1}(yu_1+wu_2)^{n-1}$$
$$= \ldots + a^{(k)}_{m,n}(-1)^{m-1}\binom{k}{m-1}u_1^{k-m+1}u_2^{m-1}+\ldots.$$

By setting $\lambda = k-m-i+2$, we have the following.

Proposition 4.5. *The (m,n)-entry of $\rho^{(k)}\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is*

$$a^{(k)}_{m,n} = (-1)^{n-m}\sum_{\lambda\in\mathbb{Z}}\binom{k-m+1}{\lambda-1}\binom{m-1}{n-\lambda}x^{k-m-\lambda+2}y^{\lambda-1}z^{m-n+\lambda-1}w^{n-\lambda}.$$

Here $a^{(k)}_{m,n}$ are polynomials of x, y, z, w with coefficients from $\mathbb{Z}$. Therefore, $\rho^{(k)}(\mathrm{SL}(2,\mathbb{Z}))\subset \mathrm{GL}(k+1,\mathbb{Z})$.

4.3 Cohomological FM transforms

We now recall some important notions from finite continued fraction theory. See [HW] for further details.

Let $\mathbf{m} = (m_1,\ldots,m_n)$ be a sequence of integers. Define s_i, t_i for $0\le i\le n$ by

$$\left.\begin{aligned} s_0=1,\quad s_1=m_1,\quad s_k=m_ks_{k-1}+s_{k-2} \\ t_0=0,\quad t_1=1,\quad t_k=m_kt_{k-1}+t_{k-2}\end{aligned}\right\}\qquad 2\le k\le n. \tag{4.3}$$

The key result for us is the following standard fact which we reproduce for the reader's convenience:

Proposition 4.6. *If we write the finite continued fraction by*

$$[m_1,m_2,\ldots,m_n] = m_1+\cfrac{1}{m_2+\cfrac{1}{\cdots+\cfrac{1}{m_n}}}$$

then $\dfrac{s_n}{s_{n-1}} = [m_n,\ldots,m_1]$, $\dfrac{t_n}{t_{n-1}} = [m_n,\ldots,m_2]$, $\dfrac{s_n}{t_n} = [m_1,\ldots,m_n]$ *and* $s_nt_{n-1}-s_{n-1}t_n = (-1)^n$.

Let (X,L) be a g-dimensional principally polarized abelian variety. The transform

$\Phi_{\mathbf{m}} : D^b(X) \to D^b(X)$ is defined by

$$\Phi_{\mathbf{m}} = \Phi \circ L^{(-1)^{n+1}m_n} \circ \Phi \circ \cdots \circ L^{-m_2} \circ \Phi \circ L^{m_1} \circ \Phi.$$

Here Φ is the FM transform from X to X with the Poincaré line bundle $\mathscr{P}$ on $X \times X$ as its kernel and L^k is $L^k \otimes (-)$.

Proposition 4.7. *The isometric automorphism associated to the FM transform $\Phi_{\mathbf{m}}$ is*

$$f_{\mathbf{m}} = (-1)^{\frac{n(n+1)}{2}} \begin{pmatrix} (-1)^{n+1}t_n & (-1)^{n+1}s_n \\ t_{n-1} & s_{n-1} \end{pmatrix}.$$

Proof. By induction on n. $\square$

Assume the Picard rank of X is one and let ℓ be $c_1(L)$. As usual, we write the Chern character $(a_0, a_1\ell, a_2\ell^2/2, \ldots, a_g\ell^g/g!)$ of any $E \in D^b(X)$ by $\mathrm{ch}(E) = (a_0, a_1, a_2, \ldots, a_g)$. So the induced transform on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ can be expressed as a $(g+1)\times(g+1)$ invertible matrix.

Example 4.8. The following examples of induced FM transforms on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ are important in this thesis. We identify them in matrix form as images of the corresponding isometric automorphisms under $\rho^{(g)}$ as given by Proposition 4.5.

$\Phi_{\mathcal{E}}$	$\Phi^{\mathrm{H}}_{\mathcal{E}}$
$[1] = \Phi^{X\to X}_{\mathcal{O}_\Delta[1]}$	$-I_{g+1} = -\rho^{(g)}\left(f_{[1]}\right)$
$\Phi_{(x,\widehat{x})} = t_{x*}(-) \otimes \mathscr{P}_{\widehat{x}}$	$I_{g+1} = \rho^{(g)}\left(f_{(x,\widehat{x})}\right)$
$\Phi = \Phi^{X\to X}_{\mathscr{P}}$	$\mathrm{Adiag}\,(1, -1, \ldots, (-1)^g) = \rho^{(g)}\,(f_{\mathscr{P}})$
$L \otimes (-) = \Phi^{X\to X}_{\delta_* L}$	$\binom{i-1}{j-1}_{1\le i,j\le g+1} = \rho^{(g)}\,(f_{\delta_* L})$

Since (X, L) is principally polarized, any FM transform $\Phi_{\mathcal{E}}$ in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\mathrm{Aut}\, D^b(X)$ is isomorphic to $\Phi_{\mathbf{m}} \circ \Phi_{(s,\widehat{s})} \circ [p]$ for some sequence of integers $\mathbf{m}$, $s \in X$, $\widehat{s} \in \widehat{X}$ and $p \in \mathbb{Z}$. The induced cohomological transform Υ^{H} on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ of $\Upsilon \in \mathrm{Aut}\, D^b(X)$ gives rise to a well defined group representation $\mathrm{Aut}\, D^b(X) \to \mathrm{GL}(H^{2*}_{\mathrm{alg}}(X, \mathbb{Q}))$, $\Upsilon \mapsto \Upsilon^{\mathrm{H}}$. Therefore, we have $\Phi^{\mathrm{H}}_{\mathcal{E}} = (-1)^p \Phi^{\mathrm{H}}_{\mathbf{m}}$, and since $\rho^{(g)}$ is a group homomorphism $\Phi^{\mathrm{H}}_{\mathbf{m}} = \rho^{(g)}(f_{\mathbf{m}})$. Also $f_{\mathcal{E}} = f_{\mathbf{m}}$. Hence, the composition

$$\Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}} \to \rho^{(g)}(f_{\mathcal{E}})$$

of maps (4.1) and (4.2) gives us the following:

Theorem 4.9. *There is a group homomorphism*

$$(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})} \to \mathrm{GL}(H^{2*}_{\mathrm{alg}}(X,\mathbb{Q})), \quad \Phi_{\mathcal{E}} \mapsto \rho^{(g)}(f_{\mathcal{E}}),$$

such that the induced cohomological FM transform $\Phi^{\mathrm{H}}_{\mathcal{E}} = \pm\rho^{(g)}(f_{\mathcal{E}})$.

For $\mathbf{m} = (m_1, \ldots, m_n)$, let

$$\left.\begin{array}{ll} x = (-1)^{\frac{(n+1)(n+2)}{2}} t_n & y = (-1)^{\frac{(n+1)(n+2)}{2}} s_n \\ z = (-1)^{\frac{n(n+1)}{2}} t_{n-1} & w = (-1)^{\frac{n(n+1)}{2}} s_{n-1} \end{array}\right\}.$$

See (4.3) for definitions of s_k and t_k. By Proposition 4.7, the induced transform on $H^{2*}_{\mathrm{alg}}(X,\mathbb{Q})$ is $\Phi^{\mathrm{H}}_{\mathbf{m}} = \rho^{(g)}\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and its (m,n)-entry is given explicitly in Proposition 4.5.

Up to shift, any non-trivial FM transform in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\mathrm{Aut}\, D^b(X)$ is isomorphic to some FM transform $\Phi_{\mathcal{E}}$ with a universal bundle $\mathcal{E}$ on $X \times X$ as the kernel (see Lemma 2.40). Therefore, $\Phi^{\mathrm{H}}_{\mathcal{E}} = \rho^{(g)}\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for some $x, y, z, w \in \mathbb{Z}$ such that $xw - yz = 1$ and $\mathrm{rk}(\mathcal{E}_{\{s\}\times X}) = \mathrm{ch}_0(\Phi_{\mathcal{E}}(\mathcal{O}_s)) = (-1)^g y^g > 0$ for any $s \in X$.

Example 4.10. For the case $g = 2$

$$\Phi^{\mathrm{H}}_{\mathcal{E}} = \begin{pmatrix} x^2 & -2xy & y^2 \\ -xz & xw + yz & -yw \\ z^2 & -2zw & w^2 \end{pmatrix}.$$

Example 4.11. For the case $g = 3$

$$\Phi^{\mathrm{H}}_{\mathcal{E}} = \begin{pmatrix} x^3 & -3x^2y & 3xy^2 & -y^3 \\ -x^2z & x^2w + 2xyz & -y^2z - 2xyw & y^2w \\ xz^2 & -yz^2 - 2xzw & xw^2 + 2yzw & -yw^2 \\ -z^3 & 3z^2w & -3zw^2 & w^3 \end{pmatrix}.$$

Since $\left(L^k \otimes (-)\right)^{\mathrm{H}} = \rho^{(g)} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$, we have

$$\begin{aligned} \mathrm{ch}^{-w\ell/y}\left(\Phi_{\mathcal{E}}(E)\right) &= e^{w\ell/y}\ \Phi_{\mathcal{E}}^{\mathrm{H}}\left(e^{x\ell/y}\,\mathrm{ch}^{x\ell/y}(E)\right) \\ &= \rho^{(g)} \begin{pmatrix} 1 & 0 \\ -w/y & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x/y & 1 \end{pmatrix} \mathrm{ch}^{x\ell/y}(E). \end{aligned}$$

Since $xw - yz = 1$, we obtain the following presentation.

Theorem 4.12.

$$\begin{aligned} \mathrm{ch}^{-w\ell/y}\left(\Phi_{\mathcal{E}}(E)\right) &= \rho^{(g)} \begin{pmatrix} 0 & y \\ -1/y & 0 \end{pmatrix} \mathrm{ch}^{x\ell/y}(E) \\ &= (-1)^g y^g \,\mathrm{Adiag}\left(1, \frac{-1}{y^2}, \ldots, \frac{(-1)^{g-1}}{y^{2(g-1)}}, \frac{(-1)^g}{y^{2g}}\right) \mathrm{ch}^{x\ell/y}(E). \end{aligned}$$

Remark 4.13. As a result of this theorem, we can see that the induced transform on $H^{2*}_{\mathrm{alg}}(X,\mathbb{Q})$ of any non-trivial FM transform in $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ with respect to the appropriate twisted Chern characters looks somewhat similar to the induced transform of Φ on $H^{2*}_{\mathrm{alg}}(X,\mathbb{Q})$ with the usual Chern characters.

Chapter 5

FM Transforms and Bridgeland Stability Conditions

This chapter discusses various connections between the Fourier-Mukai transforms and Bridgeland stability conditions on abelian varieties with special consideration for abelian threefolds in the second section.

5.1 Action of FM transforms on stability conditions

Recall that a Bridgeland stability condition on a triangulated category $\mathcal{D}$ consists of a stability function Z on the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$ with the Harder-Narasimhan property. If $\Upsilon \in \operatorname{Aut} \mathcal{D}$ and $W : K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism, then $(\Upsilon \cdot W)([E]) = W\left([\Upsilon^{-1}(E)]\right)$ defines a left action of the group $\operatorname{Aut} \mathcal{D}$ on $\operatorname{Hom}(K(\mathcal{D}), \mathbb{C})$. This action can be extended to the natural left action of $\operatorname{Aut} \mathcal{D}$ on the space of all stability conditions on $\mathcal{D}$ by defining $\Upsilon \cdot (Z, \mathcal{A}) = (\Upsilon \cdot Z, \Upsilon(\mathcal{A}))$.

Let (X, L) be a principally polarized g-dimensional abelian variety with Picard rank one and let ℓ be $c_1(L)$. The Todd class of X is trivial and so for any object in $D^b(X)$ the Mukai vector is the Chern character. Any complexified class in $\mathrm{NS}_{\mathbb{C}}(X)$ is of the form $u\ell$ for some $u = b + im \in \mathbb{C}$, where $b, m \in \mathbb{R}$. Assume $m \neq 0$. Consider the function $Z_{u\ell}$ defined by

$$Z_{u\ell}(E) = -\int_X e^{-u\ell} \operatorname{ch}(E).$$

If we denote the Mukai pairing on X by $\langle -, - \rangle$ then $Z_{u\ell}(E) = \langle e^{u\ell}, \operatorname{ch}(E) \rangle$. It is expected that $Z_{u\ell}$ is a central charge function of some stability condition on X (see Conjecture 2.26). This is already known to be true for $g = 1, 2$ completely (see Examples 2.23 and 2.25).

Let $\Phi_{\mathcal{E}}$ be a non-trivial FM transform in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\mathrm{Aut}\, D^b(X)$, with kernel the universal bundle $\mathcal{E}$ on $X \times X$. As usual, we write the Chern character of any $E \in D^b(X)$ by $\mathrm{ch}(E) = (a_0, a_1, \ldots, a_g)$. From Section 4.3 of Chapter 4, the induced transform on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ is $\Phi^{\mathrm{H}}_{\mathcal{E}} = \rho^{(g)} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for some $x, y, z, w \in \mathbb{Z}$ satisfying $xw - yz = 1$ and $(-1)^g y^g > 0$. Also $e^{k\ell} = \left(L^k \otimes (-)\right)^{\mathrm{H}} (\mathrm{ch}(\mathcal{O}_X)) = \rho^{(g)} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \mathrm{ch}(\mathcal{O}_X)$. Then

$$\Phi^{\mathrm{H}}_{\mathcal{E}}(e^{u\ell}) = \rho^{(g)} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \mathrm{ch}(\mathcal{O}_X) = \rho^{(g)} \begin{pmatrix} x - yu & y \\ z - wu & w \end{pmatrix} \mathrm{ch}(\mathcal{O}_X)$$

and from Proposition 4.5, it is equal to $(x - yu)^g e^{(-z+wu)\ell/(x-yu)}$.

By Căldăraru-Willertons' generalization of Mukai's original result for $K3$ surfaces, the cohomological FM transforms are isometries with respect to the Mukai pairing (see Lemma 2.38). Therefore, for any $E \in D^b(X)$ we have

$$\left(\Phi_{\mathcal{E}} \cdot Z_{u\ell}\right)(E) = \langle e^{u\ell} \,,\, \mathrm{ch}(\Phi^{-1}_{\mathcal{E}}(E)) \rangle = \langle \Phi^{\mathrm{H}}_{\mathcal{E}}(e^{u\ell}) \,,\, \mathrm{ch}(E) \rangle.$$

So the function $Z_{u\ell} \in \mathrm{Hom}(K(X), \mathbb{C})$ satisfies the following key relation under the action of $\mathrm{Aut}\, D^b(X)$.

Proposition 5.1. *We have* $\Phi_{\mathcal{E}} \cdot Z_{u\ell} = (x - yu)^g \mathcal{Z}_{v\ell}$ *for* $v = (-z + wu)/(x - yu)$.

Note 5.2. If $\mathcal{A}_{u\ell}$ and $\mathcal{A}_{v\ell}$ are hearts of stability conditions associated to the central charge functions $Z_{u\ell}$ and $Z_{v\ell}$ respectively, then by the above proposition one can expect $\Phi_{\mathcal{E}}(\mathcal{A}_{u\ell})$ is a tilt of $\mathcal{A}_{v\ell}$ associated to a torsion theory coming from the stability function $Z_{v\ell}$. Moreover, for the non-trivial FM transform $\Phi_{\mathcal{E}}$ (that is $y \neq 0$), when $(x - yu)^g$ is real, we would expect the equivalence (for some integer q)

$$\Phi_{\mathcal{E}}(\mathcal{A}_{u\ell}) \cong \mathcal{A}_{v\ell}[q].$$

Here $u = x/y + \lambda e^{il\pi/g}$ and $v = -w/y - \frac{1}{\lambda y^2} e^{-il\pi/g}$ for some $l \in \mathbb{Z} \setminus g\mathbb{Z}$ and $0 \neq \lambda \in \mathbb{R}$. The numerologies given for $g = 2$ and $g = 3$ cases are important in the rest of this thesis.

5.2 Relation of FM transforms to stability conditions

Let (X, L) be a principally polarized abelian threefold with Picard rank one and let ℓ be $c_1(L)$.

Let Υ be a non-trivial FM transform in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\operatorname{Aut} D^b(X)$, with kernel the universal bundle $\mathcal{E}$ on $X \times X$. Then the induced transform on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ is $\Upsilon^{\mathrm{H}} = \rho \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for some $x, y, z, w \in \mathbb{Z}$ with $xw - yz = 1$ and $y < 0$ (see Example 4.11). Here and in the rest of the thesis we write ρ for $\rho^{(3)}$. Now we have $\mathrm{ch}(\mathcal{E}_{\{s\}\times X}) = (-y^3, y^2 w, -yw^2, w^3)$. Let $\widehat{\Upsilon}$ be the FM transform with kernel $\mathsf{Swap}^*(\mathcal{E}^*)$. Then $\widehat{\Upsilon}[3]$ is the quasi-inverse of Υ and so we have $\widehat{\Upsilon}^{\mathrm{H}} = \rho \begin{pmatrix} -w & y \\ z & -x \end{pmatrix}$. Also $\mathrm{ch}(\mathcal{E}^*_{X\times\{s\}}) = (-y^3, -y^2 x, -yx^2, -x^3)$. For $g = 3$ case, Theorem 4.12 says

$$\mathrm{ch}^{-w\ell/y}\left(\Upsilon(E)\right) = \rho \begin{pmatrix} 0 & y \\ -1/y & 0 \end{pmatrix} \mathrm{ch}^{x\ell/y}(E) = \mathrm{Adiag}\left(-y^3, y, -\frac{1}{y}, \frac{1}{y^3}\right) \mathrm{ch}^{x\ell/y}(E),$$

and we have $\mathrm{ch}^{-w\ell/y}(\mathcal{E}_{\{s\}\times X}) = (-y^3, 0, 0, 0)$ and $\mathrm{ch}^{x\ell/y}(\mathcal{E}^*_{X\times\{s\}}) = (-y^3, 0, 0, 0)$.

For $\lambda \in \mathbb{Q}_{>0}$, let

$$\left.\begin{aligned} b &= \left(\frac{x}{y} + \frac{\lambda}{2}\right) & m &= \frac{\sqrt{3}\lambda}{2} \\ b' &= \left(-\frac{w}{y} - \frac{1}{2\lambda y^2}\right) & m' &= \frac{\sqrt{3}}{2\lambda y^2} \end{aligned}\right\}. \tag{5.1}$$

Recall that $Z_{\omega,B}(E) = -\int_X e^{-B-i\omega}\,\mathrm{ch}(E)$.

Proposition 5.3. *For $E \in D^b(X)$, we have the following:*

(i) if $\mathrm{ch}^{x\ell/y}(E) = (a_0, a_1, a_2, a_3)$ then $\Im Z_{m\ell,b\ell}(E) = \frac{3\sqrt{3}\lambda}{2}(a_2 - \lambda a_1)$, and

(ii) if $\mathrm{ch}^{-w\ell/y}(E) = (a_0, a_1, a_2, a_3)$ then $\Im Z_{m'\ell,b'\ell}(E) = \frac{3\sqrt{3}}{2\lambda y^2}\left(a_2 + \frac{1}{\lambda y^2} a_1\right)$.

Proof. (i) Let $\mathrm{ch}^{x\ell/y}(E) = (a_0, a_1, a_2, a_3)$. Then we have

$$\begin{aligned} Z_{m\ell,b\ell}(E) &= -\int_X e^{-\left(\frac{x}{y}+\frac{\lambda}{2}\right)\ell - i\frac{\sqrt{3}\lambda}{2}\ell}\,\mathrm{ch}(E) \\ &= -\int_X e^{-\lambda\alpha\ell}\,\mathrm{ch}^{x\ell/y}(E), && \text{where } \alpha = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{i\pi}{3}} \\ &= -\lambda^3 a_0 - 3\lambda^2\alpha^2 a_1 + 3\lambda\alpha a_2 - a_3, && \text{as } \alpha^3 = -1. \end{aligned}$$

The imaginary part of $Z_{m\ell,b\ell}(E)$ gives the required expression.

(ii) Similar to the proof of (i).

$\square$

Proposition 5.4. *For $E \in D^b(X)$, we have*

$$\Im Z_{m'\ell,b'\ell}(\Upsilon(E)) = -\frac{1}{|\lambda y|^3} \Im Z_{m\ell,b\ell}(E), \text{ and}$$
$$\Im Z_{m\ell,b\ell}(\widehat{\Upsilon}[1](E)) = -|\lambda y|^3 \Im Z_{m'\ell,b'\ell}(E).$$

Proof. Let $\mathrm{ch}^{x\ell/y}(E) = (a_0, a_1, a_2, a_3)$. From Proposition 5.3, $\Im Z_{m\ell,b\ell}(E) = \frac{3\sqrt{3}\lambda}{2}(a_2 - \lambda a_1)$. By Theorem 4.12, we have $\mathrm{ch}^{-w\ell/y}(\Upsilon(E)) = (-y^3 a_3, y a_2, -a_1/y, a_0/y^3)$. So by Proposition 5.3,

$$\Im Z_{m'\ell,b'\ell}(\Upsilon(E)) = \frac{3\sqrt{3}}{2\lambda y^2}\left(-\frac{1}{y}a_1 + \frac{1}{\lambda y^2} y a_2\right) = \frac{1}{\lambda^3 y^3} \Im Z_{m\ell,b\ell}(E).$$

The result follows as $y < 0$. Similarly one can prove the other equality. $\square$

The aim of the next chapters is to prove the following equivalences of abelian categories.

Theorem 5.5. *The FM transforms $\Upsilon[1]$ and $\widehat{\Upsilon}[2]$ give the equivalences*

$$\Upsilon[1]\left(\mathcal{A}_{m\ell,b\ell}\right) \cong \mathcal{A}_{m'\ell,b'\ell} \text{ and } \widehat{\Upsilon}[2]\left(\mathcal{A}_{m'\ell,b'\ell}\right) \cong \mathcal{A}_{m\ell,b\ell}$$

of the abelian categories.

Remark 5.6. One can see that b, m, b', m' in the above theorem are exactly the numbers given for the $g = 3$ case in Note 5.2. Moreover, the shifts are compatible with the images of the skyscraper sheaves $\mathcal{O}_s$ under the FM transforms that are minimal objects in the corresponding abelian categories, as discussed in Example 3.9.

The notion of tilt stability can be extended from rational to real as considered in [Macr2] for $\mathbb{P}^3$. As a result of the above theorem we get the following.

Theorem 5.7. *The strong Bogomolov-Gieseker type inequality holds for tilt stable objects of X with zero tilt slope.*

Proof. By [Macr2, Proposition 2.4] it is enough to consider a dense family of classes $\omega = \alpha\ell$, $B = \beta\ell$ such that $\alpha/\sqrt{3} \in \mathbb{Q}_{>0}$, $\beta \in \mathbb{Q}$. Then for given α, β one can easily find

$x, y \in \mathbb{Z}$, $\lambda \in \mathbb{Q}$ such that $\gcd(x,y) = 1$, $\alpha = \sqrt{3}\lambda/2$, $\beta = x/y + \lambda/2$. Now using the Euclidean algorithm and Proposition 4.6 (for example, see Appendix A of [BH]), one can find a non-trivial FM transform Υ which gives the equivalence of abelian categories as in Theorem 5.5. Therefore, we only need to prove the claim for objects in $\mathscr{C}_{m\ell,b\ell}$.

By Proposition 3.13, it is enough to check that the strong Bogomolov-Gieseker type inequality is satisfied by each object in $\mathscr{M}_{m\ell,b\ell} \subset \mathscr{C}_{m\ell,b\ell}$. Moreover, the objects in $\{M : M \cong \mathcal{E}^*_{X\times\{s\}}[1] \text{ for some } s \in X\} \subset \mathscr{M}_{\omega,B}$ satisfy the strong Bogomolov-Gieseker type inequality (see Example 3.9 and Note 3.10). So we only need to check the strong Bogomolov-Gieseker type inequality for objects in $\mathscr{M}_{m\ell,b\ell} \setminus \{M : M \cong \mathcal{E}^*_{X\times\{s\}}[1] \text{ for some } s \in X\}$.

Let $E \in \mathscr{M}_{m\ell,b\ell} \setminus \{M : M \cong \mathcal{E}^*_{X\times\{s\}}[1] \text{ for some } s \in X\}$. Then $E[1] \in \mathcal{A}_{m\ell,b\ell}$ is a minimal object and so by the equivalence in Theorem 5.5, $\Upsilon[1](E[1]) \in \mathcal{A}_{m'\ell,b'\ell}$ is also a minimal object. So $\Upsilon[1](E[1]) \in \mathcal{F}'_{m'\ell,b'\ell}[1]$ or $\Upsilon[1](E[1]) \in \mathcal{T}'_{m'\ell,b'\ell}$. By Proposition 5.4, $\Im Z_{m'\ell,b'\ell}(\Upsilon[1](E[1])) = 0$. Now if $\Upsilon[1](E[1]) \in \mathcal{T}'_{m'\ell,b'\ell}$ then by Lemma 2.27, $\Upsilon[1](E[1]) \in \mathrm{Coh}^0(X)$ and so E has a filtration of objects from $\{M : M \cong \mathcal{E}^*_{X\times\{s\}}[1] \text{ for some } s \in X\}$; which is not possible. So $\Upsilon[1](E) \in \mathscr{C}_{m'\ell,b'\ell}$. Moreover, for any $s \in X$ we have

$$\mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_s, \Upsilon[1](E)) \cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, \Upsilon[2](E)) \cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{E}^*_{X\times\{s\}}[1], E) = 0,$$

as $E \not\cong \mathcal{E}^*_{X\times\{s\}}[1]$. Hence, $\Upsilon[1](E) \in \mathscr{M}_{m'\ell,b'\ell}$.

Let $\mathrm{ch}^{x\ell/y}(E) = (a_0, a_1, a_2, a_3)$. Then $\Im Z_{m\ell,b\ell}(E) = 0$ implies $a_2 = \lambda a_1$ (see Proposition 5.3). Now the strong Bogomolov-Gieseker type inequality reads

$$a_3 - \lambda^2 a_1 \leq 0.$$

Let $F = \Upsilon[1](E)$ and let $F_i = H^i_{\mathrm{Coh}(X)}(F)$. From Theorem 4.12, $\mathrm{ch}^{-w\ell/y}(F) = (y^3 a_3, -y\lambda a_1, a_1/y, -a_0/y^3)$. By Proposition 3.16, we have

$$\ell^2 \,\mathrm{ch}_1^{-w\ell/y}(F_{-1}) \leq -\frac{1}{\lambda y^2}\ell^3 \,\mathrm{ch}_0^{-w\ell/y}(F_{-1}) \ \text{ and } \ \ell^2 \,\mathrm{ch}_1^{-w\ell/y}(F_0) \geq 0.$$

Therefore, $\ell^2 \,\mathrm{ch}_1^{-w\ell/y}(F) \geq -\frac{1}{\lambda y^2}\ell^3 \,\mathrm{ch}_0^{-w\ell/y}(F)$. That is $-y\lambda a_1 \geq -\frac{1}{\lambda y^2} y^3 a_3$ and so $\lambda^2 a_1 \geq a_3$ as required. $\square$

Then we can deduce the main theorem of this thesis:

Theorem 5.8. *Let α, β such that $\alpha/\sqrt{3} \in \mathbb{Q}_{>0}$ and $\beta \in \mathbb{Q}$. Then the pair $(\mathcal{A}_{\alpha\ell,\beta\ell}, Z_{\alpha\ell,\beta\ell})$ defines a Bridgeland stability condition on $D^b(X)$.*

Chapter 6

FM Transforms on Abelian Surfaces

The aim of this chapter is to apply some basic techniques of Fourier-Mukai theory to surfaces. In later chapters, we extend these results to threefolds. More specifically, we prove that any FM transform on an abelian surface gives an equivalence of two abelian categories which are double tilts of coherent sheaves. Our proofs are mainly adapted from [Yos].

6.1 Classical FM transform on abelian surface

Let (X, L) be a polarized abelian surface and let $\widehat{X}$ be the dual abelian surface. Let ℓ be $c_1(L)$ and let $\widehat{\ell}$ be the dual polarization on $\widehat{X}$.

Let Ψ be the FM transform from X to $\widehat{X}$ with kernel the Poincaré line bundle $\mathscr{P}$ on the product $X \times \widehat{X}$. Let $\widehat{\Psi}$ be the FM transform from $\widehat{X}$ to X with the same kernel on the product $\widehat{X} \times X$. Mukai proved that $\widehat{\Psi} \circ \Psi \cong (-1)^*[-2]$ and $\Psi \circ \widehat{\Psi} \cong (-1)^*[-2]$. So we have the following convergence of the spectral sequences.

Mukai Spectral Sequence 6.1.

$$E_2^{p,q} = \widehat{\Psi}^p_{\mathrm{Coh}(X)} \Psi^q_{\mathrm{Coh}(\widehat{X})}(E) \Longrightarrow H^{p+q-2}_{\mathrm{Coh}(X)}((-1)^*E),$$
$$E_2^{p,q} = \Psi^p_{\mathrm{Coh}(\widehat{X})} \widehat{\Psi}^q_{\mathrm{Coh}(X)}(E) \Longrightarrow H^{p+q-2}_{\mathrm{Coh}(\widehat{X})}((-1)^*E).$$

Notation 6.2. In this chapter we write $\widehat{\Psi}^p_{\mathrm{Coh}(X)}(E)$ and $\Psi^q_{\mathrm{Coh}(\widehat{X})}(E)$ simply by $\widehat{\Psi}^p(E)$ and $\Psi^q(E)$ respectively.

The following diagram describes the convergence of this spectral sequence for $E \in \mathrm{Coh}(X)$. See [Maci, Yos] for further details.

$$\begin{tikzcd}
\widehat{\Psi}^0\Psi^2(E) \arrow[drr, two heads, "d_2"] & & \\
\widehat{\Psi}^0\Psi^1(E) \arrow[drr, hook, "d_1"] & \widehat{\Psi}^1\Psi^1(E) & \widehat{\Psi}^2\Psi^1(E) \\
& & \widehat{\Psi}^2\Psi^0(E)
\end{tikzcd}$$

Immediately from the convergence of this spectral sequence for $E \in \mathrm{Coh}(X)$, we deduce that

- $\Psi^0(E) \in V^{\widehat{\Psi}}_{\mathrm{Coh}(X)}(2)$, and
- $\Psi^2(E) \in V^{\widehat{\Psi}}_{\mathrm{Coh}(X)}(0)$,

and there exist

- an injection $d_1 : \widehat{\Psi}^0\Psi^1(E) \hookrightarrow \widehat{\Psi}^2\Psi^0(E)$, and
- a surjection $d_2 : \widehat{\Psi}^0\Psi^2(E) \twoheadrightarrow \widehat{\Psi}^2\Psi^1(E)$.

Recall that the subcategories $\mathcal{F}_0 = \mathrm{HN}^{\mu}_{\ell,0}((-\infty, 0])$ and $\mathcal{T}_0 = \mathrm{HN}^{\mu}_{\ell,0}((0, +\infty])$ define a torsion pair on $\mathrm{Coh}(X)$ and let $\mathcal{A}_0 = \langle \mathcal{F}_0[1], \mathcal{T}_0 \rangle \subset D^b(X)$ be the corresponding tilt. Similarly there is a torsion pair $(\widehat{\mathcal{F}}_0, \widehat{\mathcal{T}}_0)$ on $\mathrm{Coh}(\widehat{X})$ with respect to the polarization $\widehat{\ell}$ on $\widehat{X}$, and let $\widehat{\mathcal{A}}_0 = \langle \widehat{\mathcal{F}}_0[1], \widehat{\mathcal{T}}_0 \rangle \subset D^b(\widehat{X})$ be the tilt of it.

The main aim in this section is to prove the following. Our proof is essentially the same as that of Yoshioka's in [Yos].

Theorem 6.3. ([Huy2], [Yos, Theorem 2.1]) *The FM transform Ψ gives an equivalence*

$$\Psi[1]\,(\mathcal{A}_0) \cong \widehat{\mathcal{A}}_0$$

of the abelian subcategories of the derived categories.

In order to prove this theorem we need the following results about cohomology sheaves of the images under the FM transforms.

Proposition 6.4. *Let $E \in \mathrm{Coh}(X)$. Then $\Psi^0(E)$ is a locally free sheaf.*

Proof. For any $s \in \widehat{X}$ we have

$$\begin{aligned}
\mathrm{Ext}^1_{\widehat{X}}(\Psi^0(E), \mathcal{O}_s) &\cong \mathrm{Hom}_X(\widehat{\Psi}\Psi^0(E), \widehat{\Psi}(\mathcal{O}_s[1])) \\
&\cong \mathrm{Hom}_X(\widehat{\Psi}^2\Psi^0(E)[-2], \mathscr{P}_{X\times\{s\}}[1]) \\
&\cong \mathrm{Hom}_X(\widehat{\Psi}^2\Psi^0(E), \mathscr{P}_{X\times\{s\}}[3]) = 0.
\end{aligned}$$

This completes the proof as required. □

Proposition 6.5. *Let $E \in \mathrm{Coh}(X)$. Then we have the following:*

(i) if $E \in \mathcal{T}_0$ then $\Psi^2(E) = 0$, and

(ii) if $E \in \mathcal{F}_0$ then $\Psi^0(E) = 0$.

Proof. (i) Let $E \in \mathcal{T}_0$. Then for any $s \in \widehat{X}$ we have

$$\begin{aligned}
\mathrm{Hom}_{\widehat{X}}(\Psi^2(E), \mathcal{O}_s) &\cong \mathrm{Hom}_{\widehat{X}}(\Psi(E)[2], \mathcal{O}_s) \\
&\cong \mathrm{Hom}_{\widehat{X}}(\Psi(E)[2], \Psi(\mathscr{P}_{X\times\{-s\}})[2]) \\
&\cong \mathrm{Hom}_X(E, \mathscr{P}_{X\times\{-s\}}) = 0,
\end{aligned}$$

as $\mathscr{P}_{X\times\{-s\}} \in \mathcal{F}_0$. Therefore, $\Psi^2(E) = 0$.

(ii) Let $E \in \mathcal{F}_0$. We can assume E is slope stable by considering the Harder-Narasimhan and Jordan-Hölder filtrations. For generic $s \in \widehat{X}$ we have

$$\mathrm{Hom}_{\widehat{X}}(\Psi^1(E), \mathcal{O}_s[1]) = \mathrm{Hom}_{\widehat{X}}(\Psi^1(E), \mathcal{O}_s[2]) = \mathrm{Hom}_{\widehat{X}}(\Psi^2(E), \mathcal{O}_s[2]) = 0.$$

Hence, by applying the functor $\mathrm{Hom}_{\widehat{X}}(-, \mathcal{O}_s)$ to the distinguished triangle $\Psi^0(E) \to \Psi(E) \to \tau_{\geq 1}\Psi(E) \to \Psi^0(E)[1]$ in $D^b(\widehat{X})$, for generic $s \in \widehat{X}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\widehat{X}}(\Psi^0(E), \mathcal{O}_s) &\cong \mathrm{Hom}_{\widehat{X}}(\Psi(E), \mathcal{O}_s) \\
&\cong \mathrm{Hom}_{\widehat{X}}(\Psi(E), \Psi(\mathscr{P}_{X\times\{-s\}})[2]) \\
&\cong \mathrm{Hom}_X(E, \mathscr{P}_{X\times\{-s\}}[2])
\end{aligned}$$

$$\cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_{X\times\{-s\}}, E)^{*}.$$

If $\mu_{\ell,0}(E) < 0$ then $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_{X\times\{-s\}}, E) = 0$. Otherwise $\mu_{\ell,0}(E) = 0$ and since E is assumed to be slope stable, any map in $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_{X\times\{-s\}}, E)$ must be an isomorphism and so $\Psi^0(E) = 0$.

Therefore, for generic $s \in \widehat{X}$, $\mathrm{Hom}_{\widehat{\mathrm{X}}}(\Psi^0(E), \mathcal{O}_s) = 0$. By Proposition 6.4, if $\Psi^0(E) \neq 0$ then it is locally free, and so we have $\Psi^0(E) = 0$.

□

Proposition 6.6. *Let $E \in \mathrm{Coh}(X)$. Then*

(i) $\Psi^2(E) \in \widehat{\mathcal{T}}_0$*, and*

(ii) $\Psi^0(E) \in \widehat{\mathcal{F}}_0$.

Proof. (i) By the Harder-Narasimhan filtration of $\Psi^2(E)$ there exists $T \in \widehat{\mathcal{T}}_0$ and $F \in \widehat{\mathcal{F}}_0$ such that

$$0 \to T \to \Psi^2(E) \to F \to 0$$

is a short exact sequence in $\mathrm{Coh}(\widehat{X})$. Now apply the FM transform $\widehat{\Psi}$ to this short exact sequence and then consider the long exact sequence of $\mathrm{Coh}(X)$ cohomologies. By the Mukai Spectral Sequence 6.1, $\Psi^2(E) \in V^{\widehat{\Psi}}_{\mathrm{Coh}(X)}(0)$ and by Proposition 6.5, $\widehat{\Psi}^2(T) = \widehat{\Psi}^0(F) = 0$. Therefore, we have $\widehat{\Psi}^i(F) = 0$ for $i = 0, 1, 2$ and so $F = 0$. So $\Psi^2(E) \cong T \in \widehat{\mathcal{T}}_0$ as required.

(ii) Similar to the proof of (i).

□

Proposition 6.7. *Let $E \in \mathrm{Coh}(X)$. Then we have the following:*

(i) if $E \in \mathcal{T}_0$ *then* $\Psi^1(E) \in \widehat{\mathcal{T}}_0$*, and*

(ii) if $E \in \mathcal{F}_0$ *then* $\Psi^1(E) \in \widehat{\mathcal{F}}_0$.

Proof. (i) Let $E \in \mathcal{T}_0$. By the Harder-Narasimhan filtration of $\Psi^1(E)$ there exists $T \in \widehat{\mathcal{T}}_0$ and $F \in \widehat{\mathcal{F}}_0$ such that

$$0 \to T \to \Psi^1(E) \to F \to 0$$

is a short exact sequence in $\mathrm{Coh}(\widehat{X})$. Assume $F \neq 0$ for a contradiction. Now apply the FM transform $\widehat{\Psi}$ to this short exact sequence and then consider the long exact sequence of $\mathrm{Coh}(X)$ cohomologies. By Proposition 6.5, $\Psi^2(E) = 0$. So by the Mukai Spectral Sequence 6.1 for E, $\widehat{\Psi}^2\Psi^1(E) = 0$ and since $\widehat{\Psi}^1\Psi^1(E)$ is quotient of $E \in \mathcal{T}_0$, we have $\widehat{\Psi}^1\Psi^1(E) \in \mathcal{T}_0$. By Proposition 6.5, $\widehat{\Psi}^2(T) = 0$ and so there is a surjection $\widehat{\Psi}^1\Psi^1(E) \twoheadrightarrow \widehat{\Psi}^1(F)$. Therefore, $c_1(\widehat{\Psi}^1(F)) \cdot \ell \geq 0$, where the equality holds when $\widehat{\Psi}^1(F) \in \mathrm{Coh}^0(X)$. Also $F \in V^{\widehat{\Psi}}_{\mathrm{Coh}(X)}(1)$ and so $c_1(\widehat{\Psi}^1(F)) \cdot \ell \leq 0$. Therefore, $\widehat{\Psi}^1(F) \in \mathrm{Coh}^0(X)$. But this is not possible as $F \in V^{\widehat{\Psi}}_{\mathrm{Coh}(X)}(1)$. This is the required contradiction to complete the proof.

(ii) Similar to the proof of (i).

□

By Propositions 6.5, 6.6 and 6.7, we have the following table of results for the images under the FM transform Ψ.

E	$\Psi^0(E)$	$\Psi^1(E)$	$\Psi^2(E)$
$\mathcal{F}_0$	0	$\widehat{\mathcal{F}}_0$	$\widehat{\mathcal{T}}_0$
$\mathcal{T}_0$	$\widehat{\mathcal{F}}_0$	$\widehat{\mathcal{T}}_0$	0

Therefore, $\Psi[1]\,(\mathcal{F}_0[1]) \subset \widehat{\mathcal{A}}_0$ and $\Psi[1]\,(\mathcal{T}_0) \subset \widehat{\mathcal{A}}_0$. Similar results hold for $\widehat{\Psi}$. Since $\widehat{\Psi}[1] \circ \Psi[1] \cong (-1)^*$ and $\Psi[1] \circ \widehat{\Psi}[1] \cong (-1)^*$ we obtain the equivalence

$$\Psi[1]\,(\mathcal{A}_0) \cong \widehat{\mathcal{A}}_0$$

of the abelian subcategories of the derived categories as claimed in Theorem 6.3.

6.2 General FM transforms on abelian surface

Let (X, L) be a principally polarized abelian surface with Picard rank one and let $\ell = c_1(L)$.

Let Υ be a non-trivial FM transform in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\mathrm{Aut}\, D^b(X)$ with kernel the universal bundle $\mathcal{E}$ on $X \times X$ (see Section 4.3). Then the induced transform on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ is $\Upsilon^{\mathrm{H}} = \rho^{(2)} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for some $x, y, z, w \in \mathbb{Z}$ with

$xw - yz = 1$ (see Example 4.10). Let $\widehat{\Upsilon}$ be the FM transform with kernel $\mathsf{Swap}^*(\mathcal{E}^*)$. Then $\widehat{\Upsilon}[2]$ is the quasi-inverse of Υ and so we have $\widehat{\Upsilon}^{\mathrm{H}} = \rho^{(2)} \begin{pmatrix} -w & y \\ z & -x \end{pmatrix}$. For the $g = 2$ case, Theorem 4.12 says

$$\mathrm{ch}^{-w\ell/y}\left(\Upsilon(E)\right) = \begin{pmatrix} 0 & 0 & y^2 \\ 0 & -1 & 0 \\ 1/y^2 & 0 & 0 \end{pmatrix} \mathrm{ch}^{x\ell/y}(E),$$

and we have $\mathrm{ch}^{-w\ell/y}(\mathcal{E}_{\{s\}\times X}) = (y^2, 0, 0)$ and $\mathrm{ch}^{x\ell/y}(\mathcal{E}^*_{X\times\{s\}}) = (y^2, 0, 0)$.

Notation 6.8. (1) For E, $\Upsilon^i(E) = \Upsilon^i_{\mathrm{Coh}(X)}(E)$.

(2) For $q \in \mathbb{Q}$, $\mathcal{F}_q = \mathrm{HN}^{\mu}_{\ell,q\ell}((-\infty, 0])$, $\mathcal{T}_q = \mathrm{HN}^{\mu}_{\ell,q\ell}((0, +\infty])$, and $\mathcal{A}_q = \langle \mathcal{F}_q[1], \mathcal{T}_q \rangle$.

The following proposition generalizes the results in the first section for the abelian surface X.

Proposition 6.9. *We have the following:*

(1) for $E \in \mathrm{Coh}(X)$

(i) $\Upsilon^0(E)$ *is locally free,*

(ii) $\Upsilon^2(E) \in \mathcal{T}_{-w/y}$,

(iii) $\Upsilon^0(E) \in \mathcal{F}_{-w/y}$;

(2) for $E \in \mathcal{T}_{x/y}$

(i) $\Upsilon^2(E) = 0$,

(ii) $\Upsilon^1(E) \in \mathcal{T}_{-w/y}$;

(3) for $E \in \mathcal{F}_{x/y}$

(i) $\Upsilon^0(E) = 0$,

(ii) $\Upsilon^1(E) \in \mathcal{F}_{-w/y}$.

Proof. Proofs are identical to the corresponding Propositions 6.4, 6.5, 6.6 and 6.7 in the previous section after replacing the Chern characters with their twisted counterparts. □

We have the following table of results for the images under the FM transforms Υ and $\widehat{\Upsilon}$.

E	$\Upsilon^0(E)$	$\Upsilon^1(E)$	$\Upsilon^2(E)$
$\mathcal{F}_{x/y}$	0	$\mathcal{F}_{-w/y}$	$\mathcal{T}_{-w/y}$
$\mathcal{T}_{x/y}$	$\mathcal{F}_{-w/y}$	$\mathcal{T}_{-w/y}$	0
E	$\widehat{\Upsilon}^0(E)$	$\widehat{\Upsilon}^1(E)$	$\widehat{\Upsilon}^2(E)$
$\mathcal{F}_{-w/y}$	0	$\mathcal{F}_{x/y}$	$\mathcal{T}_{x/y}$
$\mathcal{T}_{-w/y}$	$\mathcal{F}_{x/y}$	$\mathcal{T}_{x/y}$	0

Therefore, $\Upsilon[1]\left(\mathcal{F}_{x/y}[1]\right) \subset \mathcal{A}_{-w/y}$ and $\Upsilon[1]\left(\mathcal{T}_{x/y}\right) \subset \mathcal{A}_{-w/y}$. Also $\widehat{\Upsilon}[1]\left(\mathcal{F}_{-w/y}[1]\right) \subset \mathcal{A}_{x/y}$ and $\widehat{\Upsilon}[1]\left(\mathcal{T}_{-w/y}\right) \subset \mathcal{A}_{x/y}$. Since $\widehat{\Upsilon}[1] \circ \Upsilon[1] \cong \mathrm{id}_{D^b(X)}$ and $\Upsilon[1] \circ \widehat{\Upsilon}[1] \cong \mathrm{id}_{D^b(X)}$ we obtain the following as expected in Note 5.2 for the $g = 2$ case.

Theorem 6.10. *The FM transform Υ gives the equivalence*

$$\Upsilon[1]\left(\mathcal{A}_{x/y}\right) \cong \mathcal{A}_{-w/y}$$

of the abelian categories.

The above equivalence generalizes Yoshioka's result in [Yos, Theorem 2.1] and is more precise than the existence result in [Huy2].

Chapter 7

Classical FM Transform on Abelian Threefolds

In the rest of this thesis, (X, L) will be a principally polarized abelian threefold with Picard rank one and let ℓ be $c_1(L)$. Then $\chi(L) = \ell^3/6 = 1$ and we write the Chern character $(a_0, a_1\ell, a_2\ell^/2, a_3\ell^3/6)$ of any $E \in D^b(X)$ by (a_0, a_1, a_2, a_3).

If $E \in \mathrm{Coh}(X)$ then the slope $\mu(E)$ is defined by

$$\mu(E) = \mu_{\frac{1}{\sqrt{6}}\ell,0}(E).$$

That is $\mu(E) = a_1/a_0$ when $a_0 \neq 0$, and $\mu(E) = +\infty$ when $a_0 = 0$. Throughout this chapter we mostly use μ slope for coherent sheaves and we simply write

$$\mathrm{HN} = \mathrm{HN}^{\mu}_{\frac{1}{\sqrt{6}}\ell,0}.$$

We define the subcategories $\mathcal{T}_0 = \mathrm{HN}((0, +\infty])$ and $\mathcal{F}_0 = \mathrm{HN}((-\infty, 0])$.

7.1 FM transform on sheaves

Let Φ be the FM transform with kernel the Poincaré line bundle $\mathscr{P} = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$. The restrictions $\mathscr{P}_{\{s\}\times X}$ and $\mathscr{P}_{X\times\{s\}}$ are isomorphic to the flat line bundle $t_s^*L \otimes L^{-1}$ on X, and we simply denote it by $\mathscr{P}_s$.

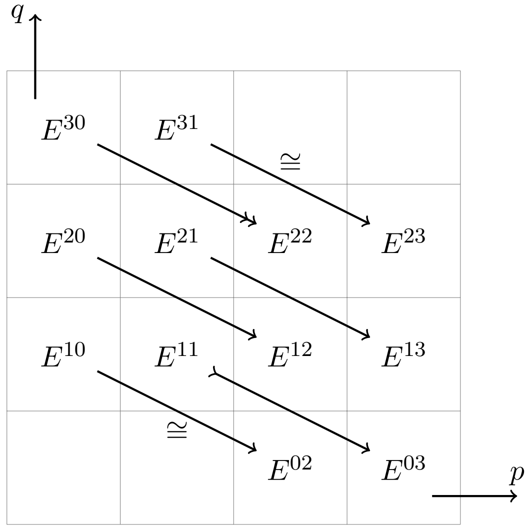
The isomorphism $\Phi \circ \Phi \cong (-1)^*[-3]$ gives us the following convergence of spectral sequence for E.

Mukai Spectral Sequence 7.1.

$$E_2^{p,q} = \Phi^p_{\mathrm{Coh}(X)} \Phi^q_{\mathrm{Coh}(X)}(E) \Longrightarrow H^{p+q-3}_{\mathrm{Coh}(X)}((-1)^* E).$$

Notation 7.2. For $E \in \mathrm{Coh}(X)$, we write $E^k = \Phi^k_{\mathrm{Coh}(X)}(E)$. Then for example $E^{120} = \Phi^0_{\mathrm{Coh}(X)} \Phi^2_{\mathrm{Coh}(X)} \Phi^1_{\mathrm{Coh}(X)}(E)$. For a line bundle L by L^k we denote the k-fold tensor product $L^{\otimes k}$ and $(L)^k = \Phi^k_{\mathrm{Coh}(X)}(L)$.

Using this notation, we can describe the second page of the Mukai Spectral Sequence for $E \in \mathrm{Coh}(X)$ as in the following diagram:



We deduce the following immediately from the Mukai Spectral Sequence:

$$E^{00} = E^{01} = E^{32} = E^{33} = 0, \;\; E^{10} \cong E^{02} \text{ and } E^{31} \cong E^{23}.$$

Moreover, we have the following by considering the convergence of the Mukai Spectral Sequence.

Proposition 7.3. *Let $E \in \mathrm{Coh}(X)$. Then we have the following:*

(i) if $E^0 = 0$ then $E^{10} = E^{11} = 0$, and

(ii) if $E^3 = 0$ then $E^{22} = E^{23} = 0$.

Let $\mathbf{R}\Delta$ denote the derived dualizing functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)[3]$. Then, due to Mukai,

$$(\Phi \circ \mathbf{R}\Delta)[3] \cong (-1)^* \mathbf{R}\Delta \circ \Phi \tag{7.1}$$

(see [Muk2, (3.8)]). This gives us the convergence of the following spectral sequence.

"Duality" Spectral Sequence 7.4.

$$\Phi^p_{\mathrm{Coh}(X)}\left(\mathcal{E}xt^{q+3}(E,\mathcal{O}_X)\right) \Longrightarrow \;?\; \Longleftarrow (-1)^*\,\mathcal{E}xt^{p+3}\left(\Phi^{3-q}_{\mathrm{Coh}(X)}(E),\mathcal{O}_X\right)$$

for $E \in \mathrm{Coh}(X)$.

See the diagram in the proof of (ii) of Proposition 7.19 for an example of the convergence of this spectral sequence.

Notation 7.5. Any $E \in \mathrm{Coh}(X)$ fits into a short exact sequence

$$0 \to T \to E \to F \to 0$$

in $\mathrm{Coh}(X)$ for some $T \in \mathcal{T}_0$ and $F \in \mathcal{F}_0$. Denote $T(E) = T$ and $F(E) = F$.

Any torsion free sheaf E fits into a short exact sequence

$$0 \to E \to E^{**} \to Q \to 0$$

in $\mathrm{Coh}(X)$ for some $Q \in \mathrm{Coh}^{\leq 1}(X)$. Here E^{**} is a reflexive sheaf. If E is rank one then E^{**} is a line bundle (see Lemma 2.46) and so $E^{**} \cong L^k\mathscr{P}_s$ for some $k \in \mathbb{Z}$ and $s \in X$.

Notation 7.6. If E is a rank one torsion free sheaf with $c_1(E) = k\ell$ then we can write $E = L^k\mathscr{P}_x\mathcal{I}_C$. Here $\mathcal{I}_C$ is the ideal sheaf of the structure sheaf

$$\mathcal{O}_C = L^{-k}\mathscr{P}_{-x} \otimes (E^{**}/E) \in \mathrm{Coh}^{\leq 1}(X)$$

of a subscheme $C \subset X$ of dimension ≤ 1.

Proposition 7.7. *Let $E \in \mathrm{Coh}(X)$. Then E^0 is a reflexive sheaf.*

Proof. Let $s \in X$. Then for $0 \leq i \leq 2$, we have

$$\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, E^0[i]) \cong \mathrm{Hom}_{\mathrm{X}}(\Phi(\mathcal{O}_s), \Phi(E^0)[i]) \cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{02}[-2+i])$$

from the convergence of the Mukai Spectral Sequence 7.1 for E. So $\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, E^0) = \mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_s, E^0) = 0$, and

$$\mathrm{Ext}^2_{\mathrm{X}}(\mathcal{O}_s, E^0) \cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{02})$$

$$\cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{10}), \quad \text{by the Mukai Spectral Sequence for } E$$
$$\cong \mathrm{Hom}_{\mathrm{X}}(\Phi(\mathcal{O}_s), \Phi(E^1))$$
$$\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, E^1).$$

Hence, as any map $\mathcal{O}_s \to E^1$ must factor through the torsion subsheaf of E^1 and E^1 is coherent, only finitely many of these can be non-zero. So $\dim \mathrm{Sing}(E^0) = \dim\{s \in X : \mathrm{Ext}^2_{\mathrm{X}}(\mathcal{O}_s, E^0) \neq 0\} \leq 0$. Therefore, E^0 is a reflexive sheaf (from Lemma 2.47). □

Proposition 7.8. *Let $E \in \mathrm{Coh}(X)$. Then we have the following:*

(i) if $E \in \mathcal{T}_0$ then $E^3 = 0$, and

(ii) if $E \in \mathcal{F}_0$ then $E^0 = 0$.

Proof. (i) Let $E \in \mathcal{T}_0$. Then for any $s \in X$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{X}}(E^3, \mathcal{O}_s) &\cong \mathrm{Hom}_{\mathrm{X}}(\Phi(E)[3], \mathcal{O}_s)\\ &\cong \mathrm{Hom}_{\mathrm{X}}(\Phi(E)[3], \Phi(\mathscr{P}_{-s})[3])\\ &\cong \mathrm{Hom}_{\mathrm{X}}(E, \mathscr{P}_{-s}) = 0,\end{aligned}$$

as $\mathscr{P}_{-s} \in \mathcal{F}_0$. Therefore, $E^3 = 0$ as required.

(ii) Let $E \in \mathcal{F}_0$. We can assume E is μ-stable using the Harder-Narasimhan and Jordan-Hölder filtrations. For generic $s \in X$ and $i = 1, 2$ we have

$$\mathrm{Hom}_{\mathrm{X}}(E^1, \mathcal{O}_s[i]) = \mathrm{Hom}_{\mathrm{X}}(E^2, \mathcal{O}_s[i+1]) = \mathrm{Hom}_{\mathrm{X}}(E^3, \mathcal{O}_s[i+2]) = 0.$$

Hence, for generic $s \in X$,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{X}}(E^0, \mathcal{O}_s) &\cong \mathrm{Hom}_{\mathrm{X}}(\Phi(E), \mathcal{O}_s)\\ &\cong \mathrm{Hom}_{\mathrm{X}}(\Phi(E), \Phi(\mathscr{P}_{-s})[3])\\ &\cong \mathrm{Hom}_{\mathrm{X}}(E, \mathscr{P}_{-s}[3])\\ &\cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_{-s}, E)^*.\end{aligned}$$

If $\mu(E) < 0$ then $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_{-s}, E) = 0$. Otherwise, $\mu(E) = 0$ and since E is assumed to be μ-stable, any map in $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_{-s}, E)$ must be an isomorphism; so $E^0 = 0$.

Therefore, for generic $s \in X$, $\mathrm{Hom}_{\mathrm{X}}(E^0, \mathcal{O}_s) = 0$. By Proposition 7.7, if $E^0 \neq 0$ then it is reflexive, and so we have $E^0 = 0$.

□

Proposition 7.9. *Let* $E \in \mathrm{Coh}(X)$. *Then*

(i) $E^3 \in \mathcal{T}_0$, *and*

(ii) $E^0 \in \mathcal{F}_0$.

Proof. (i) Let $T = T(E^3) \in \mathcal{T}_0$ and $F = F(E^3) \in \mathcal{F}_0$, so that

$$0 \to T \to E^3 \to F \to 0$$

is a short exact sequence in $\mathrm{Coh}(X)$. Now we need to show that $F = 0$. Apply Φ to the above short exact sequence and consider the long exact sequence of $\mathrm{Coh}(X)$-cohomologies. Then we have $F \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$, $T \in V^{\Phi}_{\mathrm{Coh}(X)}(0,1,2)$ and the long exact sequence

$$0 \to T^1 \to E^{31} \to F^1 \to T^2 \to 0$$

in $\mathrm{Coh}(X)$. Here $E^{31} \cong E^{23}$ (from the Mukai Spectral Sequence 7.1 for E) and so

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{X}}(E^{31}, F^1) &\cong \mathrm{Hom}_{\mathrm{X}}(E^{23}, F^1) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(\Phi(E^2)[3], \Phi(F)[1]) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(E^2, F[-2]) = 0.
\end{aligned}$$

Hence, $F \cong (-1)^* F^{12} \cong (-1)^* T^{22} = 0$ (from Proposition 7.3 for T) as required.

(ii) Similar to the proof of (i).

□

Proposition 7.10. *Let* $E \in \mathcal{F}_0$. *Then* E^1 *is a reflexive sheaf.*

Proof. By Proposition 7.8, $E^0 = 0$. Let $s \in X$. From the convergence of the Mukai Spectral Sequence 7.1 for E and $0 \leq i \leq 2$, we have

$$\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, E^1[i]) \cong \mathrm{Hom}_{\mathrm{X}}(\Phi(\mathcal{O}_s), \Phi(E^1)[i])$$

$$\cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{12}[i-2])$$

as $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, \tau_{>2}\Phi(E^1)[i]) \cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{13}[i-3]) = 0$. Therefore, $\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, E^1) = \mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_s, E^1) = 0$, and $\mathrm{Ext}^2_{\mathrm{X}}(\mathcal{O}_s, E^1) \cong \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{12})$.

From the convergence of the Mukai Spectral Sequence 7.1 for E,

$$0 \to E^{20} \to E^{12} \to F \to 0$$

is a short exact sequence in $\mathrm{Coh}(X)$. Here F is a subobject of $(-1)^*E$ and so $F \in \mathcal{F}_0$. By applying the functor $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, -)$, we obtain the exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{20}) \to \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{12}) \to \mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, F) \to \cdots .$$

Here $F \in \mathcal{F}_0$, and by Proposition 7.9, E^{20} is also in $\mathcal{F}_0$. Therefore, $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, F) \neq 0$ or $\mathrm{Hom}_{\mathrm{X}}(\mathscr{P}_s, E^{20}) \neq 0$ for at most a finite number of points $s \in X$. That is, $\dim \mathrm{Sing}(E^1) = \dim\{s \in X : \mathrm{Ext}^2_{\mathrm{X}}(\mathcal{O}_s, E^1) \neq 0\} \leq 0$. Therefore, E^1 is a reflexive sheaf (from Lemma 2.47). $\square$

Proposition 7.11. *If E is a torsion sheaf then $E^2 \in \mathcal{T}_0$.*

Proof. Let $T = T(E^2)$ and $F = F(E^2)$. Then $0 \to T \to E^2 \to F \to 0$ is a short exact sequence in $\mathrm{Coh}(X)$. By applying Φ, we obtain the long exact sequence

$$0 \to T^1 \to E^{21} \to F^1 \to T^2 \to 0$$

in $\mathrm{Coh}(X)$. Here $F \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$. From the convergence of the Mukai Spectral Sequence 7.1 for E, E^{21} fits into the short exact sequence

$$0 \to Q \to E^{21} \to E^{13} \to 0$$

in $\mathrm{Coh}(X)$, where Q is a quotient of $(-1)^*E$. So Q is a torsion sheaf and $\mathrm{Hom}_{\mathrm{X}}(Q, F^1) = 0$ as F^1 is a reflexive sheaf (see Proposition 7.10). Therefore,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{X}}(E^{21}, F^1) &\cong \mathrm{Hom}_{\mathrm{X}}(E^{13}, F^1) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(\Phi(E^1)[3], \Phi(F)[1]) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(E^1, F[-2]) = 0.
\end{aligned}$$

Hence, $F^1 \cong T^2$, and so $F \cong (-1)^* F^{12} \cong (-1)^* T^{22} = 0$ (from Proposition 7.3 for T) as required. □

Note 7.12. For $z \in X$, let $L_z = L\mathscr{P}_z$ and the divisor D_z be the zero locus of the unique section of L_z. Since $t_z^* L \otimes L^{-1} = \mathscr{P}_z$, we have $D_z = t_z^* D_e$, where $e \in X$ is the identity element. For positive integer m, let mD_z be the divisor in the linear system $|m\ell|$. So mD_z is the zero locus of a section of the line bundle L_z^m, and we have the short exact sequence

$$0 \to L_z^{-m} \to \mathcal{O}_X \to \mathcal{O}_{mD_z} \to 0$$

in $\mathrm{Coh}(X)$. Let $E \in \mathrm{Coh}(X)$. Apply the functor $E \overset{\mathbf{L}}{\otimes} (-)$ to the above short exact sequence and consider the long exact sequence of $\mathrm{Coh}(X)$-cohomologies. Since $L_z^{-m}, \mathcal{O}_X$ are locally free, we have the long exact sequence

$$0 \to \mathrm{Tor}_1(E, \mathcal{O}_{mD_z}) \to L_z^{-m} E \to E \to E \otimes \mathcal{O}_{mD_z} \to 0$$

in $\mathrm{Coh}(X)$ and $\mathrm{Tor}_i(E, \mathcal{O}_{mD_z}) = 0$ for $i \geq 2$.

Assume $E \in \mathrm{Coh}^k(X)$ for some $k \in \{0,1,2,3\}$. For generic $z \in X$, we have $\dim(\mathrm{Supp}(E) \cap D_z) \leq (k-1)$ and so $\mathrm{Tor}_1(E, \mathcal{O}_{mD_z}) \in \mathrm{Coh}^{\leq k-1}(X)$. However, $L_z^{-m} E \in \mathrm{Coh}^k(X)$, and so $\mathrm{Tor}_1(E, \mathcal{O}_{mD_z}) = 0$. Therefore we have the short exact sequence

$$0 \to L_z^{-m} E \to E \to E|_{mD_z} \to 0 \tag{7.2}$$

in $\mathrm{Coh}(X)$. Since any $E \in \mathrm{Coh}(X)$ is an extension of sheaves from $\mathrm{Coh}^k(X)$, for generic $z \in X$ we have $\mathrm{Tor}_i(E, \mathcal{O}_{mD_z}) = 0$ for $i \geq 1$ and so the short exact sequence (7.2). Moreover, when $0 \to E_1 \to E_2 \to E_3 \to 0$ is a short exact sequence in $\mathrm{Coh}(X)$, for generic $z \in X$ we have $\mathrm{Tor}_i(E_j, \mathcal{O}_{mD_z}) = 0$ for $i \geq 1$ and all j, and so

$$0 \to E_1|_{mD_z} \to E_2|_{mD_z} \to E_3|_{mD_z} \to 0$$

is a short exact sequence in $\mathrm{Coh}(X)$.

Proposition 7.13. *Let $E \in \mathrm{Coh}^{\leq 1}(X)$. Then $E^1 \in \mathcal{T}_0$.*

Proof. Sheaf $E \in \mathrm{Coh}^{\leq 1}(X)$ fits into the torsion sequence $0 \to E_0 \to E \to E_1 \to 0$, where $E_0 \in \mathrm{Coh}^0(X)$ and $E_1 \in \mathrm{Coh}^1(X)$. Here $E_0 \in V^{\Phi}_{\mathrm{Coh}(X)}(0)$ and so $E^1 = E_1^1$. Therefore, we only need to prove the claim for a pure dimension 1 torsion sheaf E.

Pick $s \in X$ such that $\dim(\mathrm{Supp}(E) \cap D_s) \le 0$. Then for $n > 0$, we have the short exact sequence

$$0 \to L_s^{-n} E \to E \to E|_{nD_s} \to 0$$

in $\mathrm{Coh}(X)$, where $E|_{nD_s} \in \mathrm{Coh}^0(X)$. For sufficiently large $n > 0$, $L_s^{-n} E \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$ and $(L_s^{-n} E)^1 \twoheadrightarrow E^1$. Therefore, we only need to show $(L_s^{-n} E)^1 \in \mathcal{T}_0$. Let us show this by proving the claim for pure dimension one torsion sheaf $E \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$. Then $\mathrm{ch}(E) = (0, 0, \alpha, \beta)$ with some $\alpha > 0$ and $\beta \le 0$ since $\beta = -\,\mathrm{rk}(E^1)$.

Let $T = T(E^1)$ and $F = F(E^1)$. Then $0 \to T \to E^1 \to F \to 0$ is a short exact sequence in $\mathrm{Coh}(X)$. Now we need to show $F = 0$. So suppose $F \neq 0$ for a contradiction. Apply the FM transform Φ and consider the long exact sequence of $\mathrm{Coh}(X)$-cohomologies. Then we have $T \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$, $F \in V^{\Phi}_{\mathrm{Coh}(X)}(1, 2)$ and

$$0 \to F^1 \to T^2 \to (-1)^* E \to F^2 \to 0$$

is a long exact sequence in $\mathrm{Coh}(X)$.

Case (i) The map $T^2 \to (-1)^* E$ is zero:

Then $T \cong (-1)^* T^{21} \cong (-1)^* F^{11} = 0$ from the Mukai Spectral Sequence 7.1 as $F \in V^{\Phi}_{\mathrm{Coh}(X)}(1, 2)$. So $(-1)^* E \cong F^2$ and hence $F \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$. Therefore, $F \cong E^1$ and so $\mathrm{ch}(F) = (-\beta, \alpha, 0, 0)$. Here $\alpha > 0$ and which is not possible as $\mu(F) \le 0$.

Case (ii) The map $T^2 \to (-1)^* E$ is non-zero:

Let $K = \mathrm{im}(T^2 \to (-1)^* E)$. Then $K \in \mathrm{Coh}^1(X)$ and the short exact sequence $0 \to F^1 \to T^2 \to K \to 0$ in $\mathrm{Coh}(X)$ corresponds to an element from $\mathrm{Ext}^1_{\mathrm{X}}(K, F^1)$. By Proposition 7.10 F^1 is a reflexive sheaf, and so by Lemma 2.46 there exists a locally free sheaf U and a torsion free sheaf V such that $0 \to F^1 \to U \to V \to 0$ is a short exact sequence in $\mathrm{Coh}(X)$. By applying the functor $\mathrm{Hom}_{\mathrm{X}}(K, -)$, we obtain the following exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathrm{X}}(K, V) \to \mathrm{Ext}^1_{\mathrm{X}}(K, F^1) \to \mathrm{Ext}^1_{\mathrm{X}}(K, U) \to \cdots .$$

Here $\mathrm{Hom}_{\mathrm{X}}(K, V) = 0$ and $\mathrm{Ext}^1_{\mathrm{X}}(K, U) \cong \mathrm{Ext}^2_{\mathrm{X}}(U, K)^* \cong H^2(X, U^* \otimes K)^* = 0$ as $K \in \mathrm{Coh}^{\le 1}(X)$. So $\mathrm{Ext}^1_{\mathrm{X}}(K, F^1) = 0$ implies $T^2 \cong F^1 \oplus K$. Here $T^2 \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$ implies $F^1 = 0$ and so $K \cong T^2$. Then $F^2 \cong E/T^2$ and also $F \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$. Since

$F^2 \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$, it is a pure dimension one torsion sheaf. So $\mathrm{ch}(F^2) = (0, 0, \alpha', \beta')$ with some $\alpha' > 0$ and $\beta' \leq 0$. Therefore, $\mathrm{ch}(F) = (-\beta', \alpha', 0, 0)$. This is not possible as $\mu(F) \leq 0$ implies $\alpha' \leq 0$.

Therefore, $F = 0$ as required to complete the proof. $\square$

Recall from Lemma 2.43, for any positive integer s, the semi-homogeneous bundle $(L^s)^0$ is slope stable. In the rest of this section we abuse notation to write $(L^s)^0$ for the functor $(L^s)^0 \otimes (-)$.

Proposition 7.14. *Let $E_n \in \mathrm{HN}([0, +\infty))$, $n \in \mathbb{N}$ be a sequence of coherent sheaves on X. For any $s > 0$ there is $N(s) > 0$ such that for any $n > N(s)$ we have $(L^s)^0 E_n \in V^{\Phi}_{\mathrm{Coh}(X)}(3)$. Then $\mu^+(E_n) \to 0$ as $n \to +\infty$.*

Proof. Let s be a positive integer. Then by Proposition 7.8, for $n > N(s)$ we have $(L^s)^0 E_n \in \mathrm{HN}((-\infty, 0])$. Since $(L^s)^0$ is slope stable with $\mu = -1/s$, for $n > N(s)$

$$E_n \in \mathrm{HN}([0, \frac{1}{s}]).$$

Therefore, the claim in the Proposition follows by considering large enough s. $\square$

Let s be a positive integer. Consider the FM functor defined by

$$\Pi = \Phi \circ (L^s)^0 \circ \Phi[3].$$

Then $\Pi^i_{\mathrm{Coh}(X)}(\mathcal{O}_x) = 0$ for $i \neq 0$ and $\Pi^0_{\mathrm{Coh}(X)}(\mathcal{O}_x) = L^s \mathscr{P}_y$ for some $y \in X$. Define the FM functor

$$\widehat{\Pi} = \Phi \circ (L^{-s})^3 \circ \Phi.$$

One can show that $\widehat{\Pi}[3]$ is right and left adjoint to Π (and vice versa). We have $\widehat{\Pi}^i_{\mathrm{Coh}(X)}(\mathcal{O}_x) = 0$ for $i \neq 0$, and $\widehat{\Pi}^0_{\mathrm{Coh}(X)}(\mathcal{O}_x) = L^{-s} \mathscr{P}_z$ for some $z \in X$. Therefore, Π is an FM functor with kernel a locally free sheaf $\mathcal{U}$ on $X \times X$.

We have the spectral sequence

$$\Phi^p_{\mathrm{Coh}(X)} \left((L^s)^0 \, \Phi^q_{\mathrm{Coh}(X)}(E) \right) \Longrightarrow \Pi^{p+q-3}_{\mathrm{Coh}(X)}(E) \tag{7.3}$$

for E.

Proposition 7.15. *Let $E \in \mathrm{Coh}^1(X)$. Then $\mu^+((L^{-n}E)^1) \to 0$ as $n \to +\infty$.*

Proof. Since $E \in \mathrm{Coh}^1(X)$, for sufficiently large $n > 0$, we have $L^{-n}E \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$. By Proposition 7.13, $(L^{-n}E)^1 \in \mathcal{T}_0$. Let s be a positive integer. Consider the convergence of the Spectral Sequence (7.3). For large enough $n > 0$, we also have $L^{-n}E \in V^{\Pi}_{\mathrm{Coh}(X)}(1)$. Therefore, $(L^s)^0(L^{-n}E)^1 \in V^{\Phi}_{\mathrm{Coh}(X)}(3)$, and so the claim follows from Proposition 7.14. $\square$

Proposition 7.16. *Let E be a reflexive sheaf. Then for sufficiently large $n > 0$,*

(i) $L^{-n}E \in V^{\Phi}_{\mathrm{Coh}(X)}(2,3)$, *and*

(ii) $(L^{-n}E)^2 \cong (T_0)^0$ *for some* $T_0 \in \mathrm{Coh}^0(X)$.

Proof. (i) Consider a minimal locally free resolution of E,

$$0 \to F_2 \to F_1 \to E \to 0.$$

By applying the FM transform ΦL^{-n} for sufficiently large $n > 0$, we obtain $L^{-n}E \in V^{\Phi}_{\mathrm{Coh}(X)}(2,3)$.

(ii) Since E is a reflexive sheaf, there is a locally free sheaf P and a torsion free sheaf Q such that

$$0 \to E \to P \to Q \to 0$$

is a short exact sequence in $\mathrm{Coh}(X)$ (see Lemma 2.46). By applying the FM transform ΦL^{-n} for sufficiently large n we have $(L^{-n}E)^2 \cong (L^{-n}Q)^1$.

The torsion free sheaf Q fits into its structure sequence $0 \to Q \to Q^{**} \to T \to 0$ for some $T \in \mathrm{Coh}^{\leq 1}(X)$. Apply the FM transform ΦL^{-n} for sufficiently large n and consider the long exact sequence of $\mathrm{Coh}(X)$-cohomologies. Since $L^{-n}Q^{**} \in V^{\Phi}_{\mathrm{Coh}(X)}(2,3)$, we have $(L^{-n}Q)^1 \cong (L^{-n}T)^0$. The torsion sheaf $T \in \mathrm{Coh}^{\leq 1}(X)$ fits into short exact sequence $0 \to T_0 \to T \to T_1 \to 0$ in $\mathrm{Coh}(X)$ for $T_i \in \mathrm{Coh}^i(X)$, $i = 0, 1$. Therefore, $(L^{-n}T)^0 \cong (T_0)^0$, and so $(L^{-n}E)^2 \cong (T_0)^0$ as required.

$\square$

Proposition 7.17. *Let $E \in \mathrm{Coh}^1(X)$ with $E \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$. If $0 \neq T \in \mathrm{HN}([0, +\infty])$ is a subsheaf of E^1 then $\ell\,\mathrm{ch}_2(T) \leq 0$.*

Proof. Recall that for $z \in X$, D_z is the divisor of $L_z = L\mathscr{P}_z$. Choose $z \in X$ such that

$\dim(\mathrm{Supp}(E) \cap D_z) \le 0$. Then for $n > 0$, we have the short exact sequence

$$0 \to L_z^{-n} E \to E \to T_0 \to 0$$

in $\mathrm{Coh}(X)$, where $T_0 = E|_{nD_z} \in \mathrm{Coh}^0(X)$.

By applying the FM transform Φ we get the following commutative diagram for some $A \in \mathrm{HN}([0, +\infty])$.

$$\begin{tikzcd}
0 \arrow[r] & T_0^0 \arrow[r] & (L_z^{-n}E)^1 \arrow[r] & E^1 \arrow[r] & 0 \\
0 \arrow[r] & T_0^0 \arrow[r] \arrow[u, equal] & A \arrow[r] \arrow[u, hook] & T \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

We have $\mathrm{ch}_k(A) = \mathrm{ch}_k(T)$ for $k = 1, 2, 3$.

Let G be a slope semistable Harder-Narasimhan factor of A. Then, from the usual Bogomolov-Gieseker inequality, we have

$$\begin{aligned} 2\ell \,\mathrm{ch}_2(G) &\le \frac{\ell \,\mathrm{ch}_1(G)^2}{\mathrm{ch}_0(G)} \\ &\le \ell^2 \,\mathrm{ch}_1(A)\, \mu(G) \\ &\le \ell^2 \,\mathrm{ch}_1(T)\, \mu^+((L_z^{-n}E)^1). \end{aligned}$$

By Proposition 7.15, $\mu^+((L_z^{-n}E)^1) \to 0$ as $n \to +\infty$. So choose large enough $n > 0$ such that $\ell^2 \,\mathrm{ch}_1(T)\mu^+((L_z^{-n}E)^1) < \ell^3$. Since $2\ell \,\mathrm{ch}_2(G) \in \ell^3\mathbb{Z}$ we have $\ell \,\mathrm{ch}_2(G) \le 0$. Therefore, $\ell \,\mathrm{ch}_2(T) = \ell \,\mathrm{ch}_2(A) \le 0$. □

Let E be a reflexive sheaf on X. Therefore, $\dim \mathrm{Sing}(E) \le 0$, and so for generic $z \in X$ we have $\mathrm{Sing}(E) \cap D_z = \emptyset$.

Proposition 7.18. *Let m be any positive integer. Then for large enough $n > 0$,*

$$L^{-n}E|_{mD_z} \in V^{\Phi}_{\mathrm{Coh}(X)}(2).$$

Proof. The dual sheaf E^* is also reflexive (see Lemma 2.46). Consider a minimal locally free resolution of E^*:

$$0 \to G \to F \to E^* \to 0.$$

By applying the dualizing functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)$ to this short exact sequence, we get the following long exact sequence in $\mathrm{Coh}(X)$:

$$0 \to E \to F^* \to G^* \to \mathcal{E}xt^1(E^*, \mathcal{O}_X) \to 0.$$

Let $Q = \mathrm{coker}(E \to F^*)$. Since E is reflexive,

$$\mathrm{Sing}(E) = \mathrm{Sing}(E^*) = \mathrm{Supp}(\mathcal{E}xt^1(E^*, \mathcal{O}_X)).$$

By choice $\mathrm{Sing}(E) \cap D_z = \emptyset$, and so from the short exact sequence $0 \to Q \to G^* \to \mathcal{E}xt^1(E^*, \mathcal{O}_X) \to 0$, $Q|_{mD_z} \cong G^*|_{mD_z}$. So we have the short exact sequence

$$0 \to E|_{mD_z} \to F^*|_{mD_z} \to G^*|_{mD_z} \to 0$$

in $\mathrm{Coh}(X)$. Since F^* and G^* are locally free, for large enough $n > 0$ we have $L^{-n}F^*|_{mD_z}, L^{-n}G^*|_{mD_z} \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$ and so $L^{-n}E|_{mD_z} \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$. $\square$

Proposition 7.19. *We have the following:*

(i) Let $E \in \mathcal{F}_0$ be a reflexive sheaf. If $T \in \mathcal{T}_0$ is a non-trivial subsheaf of E^1 then $\ell\,\mathrm{ch}_2(T) \leq 0$.

(ii) Let $E \in \mathcal{T}_0$ be a torsion free sheaf. If $F \in \mathcal{F}_0$ is a non-trivial quotient of E^2 then $\ell\,\mathrm{ch}_2(F) \leq 0$.

Proof. (i) Since E is reflexive, $\dim \mathrm{Sing}(E) \leq 0$. Choose $x, y \in X$ such that

- $\dim(D_x \cap D_y) = 1$,
- $\mathrm{Sing}(E) \cap D_x = \emptyset$, and
- $\mathrm{Sing}(E) \cap D_y = \emptyset$.

Since E is a reflexive sheaf, Proposition 7.16 implies, for sufficiently large $m > 0$ $L_x^{-m}E \in V^{\Phi}_{\mathrm{Coh}(X)}(2,3)$. By applying the FM transform Φ to the short exact sequence

$$0 \to L_x^{-m}E \to E \to E|_{mD_x} \to 0$$

in $\mathrm{Coh}(X)$, $E|_{mD_x} \in V^{\Phi}_{\mathrm{Coh}(X)}(1,2)$ and $E^1 \hookrightarrow (E|_{mD_x})^1$. By Proposition 7.18, for large enough $n > 0$, $L_y^{-n}E|_{mD_x} \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$. By applying the FM transform Φ

to the short exact sequence

$$0 \to L_y^{-n} E|_{mD_x} \to E|_{mD_x} \to E|_{mD_x \cap nD_y} \to 0$$

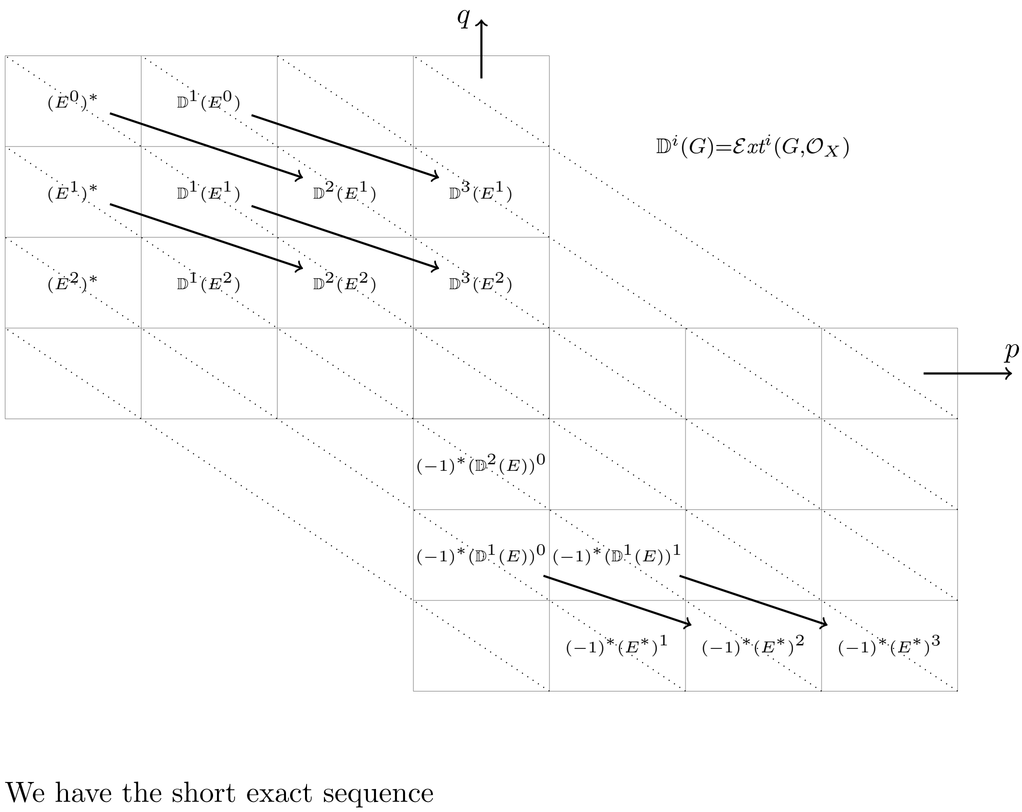
in $\mathrm{Coh}(X)$, $E|_{mD_x \cap nD_y} \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$ and $(E|_{mD_x})^1 \hookrightarrow \left(E|_{mD_x \cap nD_y}\right)^1$. Therefore, we have

$$T \hookrightarrow E^1 \hookrightarrow \left(E|_{mD_x \cap nD_y}\right)^1.$$

The result follows from Proposition 7.17.

(ii) Since $F \neq 0$ is a quotient of E^2, we have $F^* \hookrightarrow (E^2)^*$. Here $F^* \in \mathrm{HN}([0, +\infty))$ fits into short exact sequence $0 \to T \to F^* \to F_0 \to 0$ in $\mathrm{Coh}(X)$ for some $T \in \mathcal{T}_0$ and $F_0 \in \mathrm{HN}(0)$. By the usual Bogomolov-Gieseker inequality $\ell\,\mathrm{ch}_2(F_0) \leq 0$.

By Proposition 7.8, $E^3 = 0 = (E^*)^0$. Consider the convergence of the "Duality" Spectral Sequence 7.4 for E and the following diagram describes its second page.



We have the short exact sequence

$$0 \to (-1)^*(E^*)^1 \to (E^2)^* \to P \to 0$$

in $\mathrm{Coh}(X)$, for some subsheaf P of $(-1)^*(\mathcal{E}xt^1(E,\mathcal{O}_X))^0$. By Proposition 7.9, $(\mathcal{E}xt^1(E,\mathcal{O}_X))^0 \in \mathcal{F}_0$ and so $P \in \mathcal{F}_0$. Therefore, $\mathrm{Hom}_X(T,P) = 0$, and so $T \hookrightarrow (-1)^*(E^*)^1$. Here $E^* \in \mathcal{F}_0$ and so by part (i), $\ell\,\mathrm{ch}_2(T) \leq 0$. Therefore, $\ell\,\mathrm{ch}_2(F) \leq \ell\,\mathrm{ch}_2(F^{**}) = \ell\,\mathrm{ch}_2(F^*) = \ell\,\mathrm{ch}_2(F_0) + \ell\,\mathrm{ch}_2(T) \leq 0$.

□

Proposition 7.20. *For $E \in \mathrm{Coh}(X)$, we have the following:*

(i) if $E \in \mathcal{F}_0$ then $E^1 \in \mathcal{F}_0$, and

(ii) if $E \in \mathrm{HN}([0,+\infty))$ with $E^3 = 0$ then $E^2 \in \mathrm{HN}([0,+\infty])$.

Proof. (i) Assume the opposite for a contradiction. Let $T = T(E^1)$ and $F = F(E^1)$. Then $0 \to T \to E^1 \to F \to 0$ is a short exact sequence in $\mathrm{Coh}(X)$. By Proposition 7.10, E^1 is reflexive, and so the non-trivial sheaf T is torsion free. Therefore, $\ell^2\,\mathrm{ch}_1(T) > 0$. By applying the FM transform Φ to this short exact sequence, we obtain that $T \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$ and $F \in V^{\Phi}_{\mathrm{Coh}(X)}(1,2,3)$. Moreover, we have the short exact sequence

$$0 \to F^1 \to T^2 \to E_1 \to 0$$

in $\mathrm{Coh}(X)$ for some subsheaf E_1 of E^{12}. From the Mukai Spectral Sequence 7.1 for E, we have the short exact sequence

$$0 \to E^{20} \to E^{12} \to E_2 \to 0$$

in $\mathrm{Coh}(X)$ for some subsheaf E_2 of $(-1)^*E$. Therefore, $E_2 \in \mathcal{F}_0$, and by Proposition 7.9, $E^{20} \in \mathcal{F}_0$. So we have $E^{12} \in \mathcal{F}_0$. Hence, $E_1 \in \mathcal{F}_0$.

Let $T_1 = T(F^1)$ and $F_1 = F(T^2)$. They fit into the following commutative diagram for some $F_2 \in \mathcal{F}_0$.

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & F_2 \arrow[u] \arrow[r] & F_1 \arrow[u] \arrow[r] & E_1 \arrow[r] & 0 \\
0 \arrow[r] & F^1 \arrow[u] \arrow[r] & T^2 \arrow[u] \arrow[r] & E_1 \arrow[u, equal] \arrow[r] & 0 \\
 & T_1 \arrow[u] \arrow[r, equal] & T_1 \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

By Proposition 7.19, $\ell\,\mathrm{ch}_2(F_1) \leq 0$.

By applying the FM transform Φ to the short exact sequence $0 \to T_1 \to F^1 \to F_2 \to 0$ in $\mathrm{Coh}(X)$, we obtain the short exact sequence

$$0 \to F_2^1 \to T_1^2 \to F_3 \to 0$$

in $\mathrm{Coh}(X)$ for some subsheaf F_3 of F^{12}. Also $T_1 \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$. By considering the Mukai Spectral Sequence 7.1 for F, one can show $F^{12} \in \mathcal{F}_0$ and so $F_3 \in \mathcal{F}_0$. By Proposition 7.10, F_2^1 is reflexive. So T_1^2 is torsion free and it fits into short exact sequence

$$0 \to T_1^2 \to (T_1^2)^{**} \to Q \to 0$$

in $\mathrm{Coh}(X)$ for some $Q \in \mathrm{Coh}^{\leq 1}(X)$. The torsion sheaf Q fits into short exact sequence

$$0 \to Q_0 \to Q \to Q_1 \to 0$$

in $\mathrm{Coh}(X)$ for $Q_0 \in \mathrm{Coh}^0(X)$ and $Q_1 \in \mathrm{Coh}^1(X)$. By Proposition 7.16, for large enough $m > 0$, $(L^{-m}T_1^2)^1 \cong (L^{-m}Q)^0 \cong Q_0^0$. Also $(L^{-m}Q_1)^1 \cong (L^{-m}Q)^1$ and $(L^{-m}(T_1^2)^{**})^2 \cong R_0^0$ for some $R_0 \in \mathrm{Coh}^0(X)$. So we have the short exact sequence

$$0 \to (L^{-m}Q_1)^1 \to (L^{-m}T_1^2)^2 \to R_0^0 \to 0$$

in $\mathrm{Coh}(X)$. By Proposition 7.13, $(L^{-m}T_1^2)^2 \in \mathrm{HN}([0, +\infty))$. Since $R_0 \in \mathrm{Coh}^0(X)$ and $L^{-m}Q_1 \in \mathrm{Coh}^1(X) \cap V^{\Phi}_{\mathrm{Coh}(X)}(1)$, $\ell\,\mathrm{ch}_2((L^{-m}T_1^2)^2) = 0$.

The torsion free sheaf F_3 also fits into short exact sequence $0 \to F_3 \to F_3^{**} \to$

$S \to 0$ in $\mathrm{Coh}(X)$ for some $S \in \mathrm{Coh}^{\leq 1}(X)$.

Choose $x, y \in X$ such that

- $\dim(D_x \cap D_y) = 1$,
- $D_x \cap \mathrm{Supp}(Q_0) = \emptyset$,
- $\dim(\mathrm{Supp}(Q_1) \cap D_x) \leq 0$,
- $D_x \cap D_y \cap \mathrm{Supp}(Q) = \emptyset$,
- $D_x \cap D_y \cap \mathrm{Supp}(S) = \emptyset$,
- since F_2^1 is reflexive, $\mathrm{Sing}(F_2^1) \cap D_x = \mathrm{Sing}(F_2^1) \cap D_y = \emptyset$, and
- since F_3^{**} is reflexive, $\mathrm{Sing}(F_3^{**}) \cap D_x = \mathrm{Sing}(F_3^{**}) \cap D_y = \emptyset$.

From the Mukai Spectral Sequence for F_2, $F_2^1 \in V^{\Phi}_{\mathrm{Coh}(X)}(2,3)$. Since it is a reflexive sheaf, for large enough $m > 0$, $L_x^{-m} F_2^1 \in V^{\Phi}_{\mathrm{Coh}(X)}(2,3)$, and also by the choice, for large enough $n > 0$, $L_y^{-n} F_2^1|_{mD_x} \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$. So $F_2^1|_{mD_x \cap nD_y} \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$. Since $D_x \cap D_y \cap \mathrm{Supp}(S) = \emptyset$, similarly one can show $F_3|_{mD_x \cap nD_y} \cong F_3^{**}|_{mD_x \cap nD_y} \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$. Therefore, we have $T_1^2|_{mD_x \cap nD_y} \cong (T_1^2)^{**}|_{mD_x \cap nD_y} \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$.

By applying the FM transform Φ to the short exact sequence $0 \to L_x^{-m} T_1^2 \to T_1^2 \to T_1^2|_{mD_x} \to 0$ in $\mathrm{Coh}(X)$, for large enough $m > 0$, we have the long exact sequence

$$0 \to Q_0^0 \to (-1)^* T_1 \to (T_1^2|_{mD_x})^1 \to (L^{-m} T_1^2)^2 \to 0$$

in $\mathrm{Coh}(X)$. So $\left(T_1^2|_{mD_x}\right)^1 \in \mathrm{HN}([0, +\infty])$ and $\mathrm{ch}_2((T_1^2|_{mD_x})^1) = \mathrm{ch}_2(T_1)$. Moreover, we have the short exact sequence

$$0 \to T_1^2|_{mD_x} \to (T_1^2)^{**}|_{mD_x} \to Q_1|_{mD_x} \to 0$$

in $\mathrm{Coh}(X)$, where $Q_1|_{mD_x} \in \mathrm{Coh}^0(X)$. So for large enough $n > 0$, $(L^{-n} T_1^2|_{mD_x})^1 \cong (Q_1|_{mD_x})^0$.

By applying the FM transform Φ to the short exact sequence $0 \to L_y^{-n} T_1^2|_{mD_x} \to T_1^2|_{mD_x} \to T_1^2|_{mD_x \cap nD_y} \to 0$ in $\mathrm{Coh}(X)$, we have the long exact sequence

$$0 \to (Q_1|_{mD_x})^0 \xrightarrow{\alpha} (T_1^2|_{mD_x})^1 \to (T_1^2|_{mD_x \cap nD_y})^1 \to \cdots$$

in $\mathrm{Coh}(X)$. Let $T_2 = \mathrm{coker}(\alpha)$. Then $T_2 \in \mathrm{HN}([0, +\infty])$ and $\mathrm{ch}_2(T_2) = \mathrm{ch}_2(T_1)$. By Proposition 7.17, we have $\ell \,\mathrm{ch}_2(T_2) \leq 0$. So $\ell \,\mathrm{ch}_2(T^2) = \ell \,\mathrm{ch}_2(T_1) + \ell \,\mathrm{ch}_2(F_1) \leq 0$. Therefore, we have $\ell^2 \,\mathrm{ch}_1(T) \leq 0$. This is the required contradiction to complete the proof.

(ii) Since $E^* \in \mathrm{HN}((-\infty, 0])$, from (i) $(E^*)^1 \in \mathrm{HN}((-\infty, 0])$. By the convergence of the "Duality" Spectral Sequence 7.4 for E, we have $(E^2)^* \in \mathrm{HN}((-\infty, 0])$. So $E^2 \in \mathrm{HN}([0, +\infty])$ as required.

□

Corollary 7.21. *Let $E \in \mathcal{T}_0$. Then $E^2 \in \mathcal{T}_0$.*

Proof. Let $T = T(E^2)$ and $F = F(E^2)$. Then $0 \to T \to E^2 \to F \to 0$ is a short exact sequence in $\mathrm{Coh}(X)$. Now we need to show $F = 0$. Apply the FM transform Φ and consider the long exact sequence of $\mathrm{Coh}(X)$-cohomologies. So we have $F \in V^{\Phi}_{\mathrm{Coh}(X)}(1)$ and

$$0 \to T^1 \to E^{21} \to F^1 \to T^2 \to 0$$

is a long exact sequence in $\mathrm{Coh}(X)$. From the convergence of the Mukai Spectral Sequence 7.1 for E, we have the short exact sequence

$$0 \to Q \to E^{21} \to E^{13} \to 0$$

in $\mathrm{Coh}(X)$, where Q is a quotient of $(-1)^*E$. Then $Q \in \mathcal{T}_0$ and, by Proposition 7.9, $E^{13} \in \mathcal{T}_0$ and so $E^{21} \in \mathcal{T}_0$. On the other hand, by Proposition 7.20, $F^1 \in \mathcal{F}_0$. So the map $E^{21} \to F^1$ is zero and $F^1 \cong T^2$. Hence, $F \cong (-1)^* F^{12} \cong (-1)^* T^{22} = 0$ (from Proposition 7.3 for T) as required. □

7.2 Some stable reflexive sheaves

In this section we shall consider slope semistable sheaves with vanishing first and second parts of the twisted Chern characters. Such sheaves arise as the $\mathrm{Coh}(X)$-cohomology of some of the tilt-stable objects. For example, when $F \in \mathcal{B}_{\sqrt{3}q\ell, p\ell}$ is a tilt stable object with $\nu_{\sqrt{3}q\ell, p\ell}(F) = 0$ and $F_i = H^i_{\mathrm{Coh}(X)}(F)$, by Propositions 3.15 and 3.16, if $\mu_{\sqrt{3}q\ell,(p-q)\ell}(F_{-1}) = 0$ then F_{-1} is slope semistable with $\mathrm{ch}_k^{(p-q)\ell}(F_{-1}) = 0$, and if $\mu_{\sqrt{3}q\ell,(p+q)\ell}(F_0) = 0$ then F_0 is slope semistable with $\mathrm{ch}_k^{(p+q)\ell}(F_0) = 0$, for $k = 1, 2$.

First we prove the following:

Theorem 7.22. *Let E be a slope semistable sheaf with $\mathrm{ch}_k(E) = 0$ for $k = 1, 2$. Then E^{**} is a homogeneous bundle, that is E^{**} is filtered with quotients from $\mathrm{Pic}^0(X)$.*

Proof. Any torsion free sheaf E fits into the short exact sequence $0 \to E \to E^{**} \to Q \to 0$ in $\mathrm{Coh}(X)$ for some $Q \in \mathrm{Coh}^{\leq 1}(X)$. If $\mathrm{ch}_k(E) = 0$ for $k = 1, 2$ then $\ell \,\mathrm{ch}_2(E^{**}) \geq 0$ where the equality holds when $Q \in \mathrm{Coh}^0(X)$. If E is slope semistable then E^{**} is also slope semistable, and so by the usual Bogomolov-Gieseker inequality $\mathrm{ch}_2(E^{**}) = 0$.

Assume the opposite for a contradiction. Then there exists a semistable reflexive sheaf E with $\mathrm{ch}_k(E) = 0$ for $k = 1, 2$, and $H^k(X, E \otimes \mathscr{P}_x) = 0$ for $k = 0, 3$ and any $x \in X$. So we have $E^0 = E^3 = 0$. By a result of Simpson (see Lemma 2.48), we have $\mathrm{ch}_3(E) = 0$. Therefore, $\mathrm{ch}(E) = (r, 0, 0, 0)$ for some positive integer r.

By Proposition 7.20, $E^1 \in \mathrm{HN}((-\infty, 0])$, and $E^2 \in \mathrm{HN}([0, +\infty])$. So we have $\ell^2 \,\mathrm{ch}_1(E^1) \leq 0$ and $\ell^2 \,\mathrm{ch}_1(E^2) \geq 0$. Therefore, $\ell^2 \,\mathrm{ch}_1(\Phi(E)) \geq 0$ which implies $\ell \,\mathrm{ch}_2(E) \leq 0$. Since $\mathrm{ch}_2(E) = 0$, we obtain $\mathrm{ch}_1(E^1) = \mathrm{ch}_1(E^2) = 0$. Then we have

$$\mathrm{ch}(E^1) = (a, 0, -b, c), \quad \mathrm{ch}(E^2) = (a, 0, -b, -r + c),$$

for some $a > 0$ and $b \geq 0$. Moreover, we have $E^1 \in \mathrm{HN}(0)$.

If $E^{13} \neq 0$ then E^1 fits into a short exact sequence $0 \to K_1 \to E^1 \to \mathscr{P}_{z_1} \mathcal{I}_{C_1} \to 0$ in $\mathrm{Coh}(X)$. Then $K_1 \in \mathrm{HN}(0)$ and we have the following exact sequence

$$\cdots \to K_1^3 \to E^{13} \to \mathcal{O}_{-z_1} \to 0$$

in $\mathrm{Coh}(X)$. If $K_1^3 \neq 0$ then K_1 fits into a short exact sequence $0 \to K_2 \to K_1 \to \mathscr{P}_{z_2} \mathcal{I}_{C_2} \to 0$ in $\mathrm{Coh}(X)$. Then $K_2 \in \mathrm{HN}(0)$ and we have the following exact sequence

$$\cdots \to K_2^3 \to K_1^3 \to \mathcal{O}_{-z_2} \to 0$$

in $\mathrm{Coh}(X)$. We can continue this process for only a finite number of steps since $\mathrm{rk}(E^1) < +\infty$ and hence E^{13} is filtered by skyscraper sheaves. Moreover, from the convergence of the Mukai Spectral Sequence 7.1 for E, we have the short exact sequence

$$0 \to E^{20} \to E^{12} \to Q \to 0$$

in $\mathrm{Coh}(X)$, where Q is a subsheaf of $(-1)^* E$ and so $Q \in \mathrm{HN}((-\infty, 0])$. By Proposition 7.9, $E^{20} \in \mathrm{HN}((-\infty, 0])$. This implies $E^{12} \in \mathrm{HN}((-\infty, 0])$. Then $\ell^2 \,\mathrm{ch}_1(\Phi(E^1)) \leq$

0 and so $-b\ell^3 = 2\ell\,\mathrm{ch}_2(E^1) \geq 0$. Hence, $b = 0$. By Proposition 7.10, E^1 is a reflexive sheaf and since $E^1 \in \mathrm{HN}(0)$ it is slope semistable. So by Lemma 2.48, we have $c = \mathrm{ch}_3(E^1) = 0$. Therefore, $\mathrm{ch}(\Phi(E^1)) = (0,0,0,-a)$. Since $E^{13} \in \mathrm{Coh}^0(X)$, we have $\mathrm{ch}_k(E^{12}) = 0$ for $k = 0,1,2$. So $E^{12} \in \mathrm{HN}((0,+\infty])$. Therefore, $E^{12} = 0$ and we have the short exact sequence

$$0 \to (-1)^*E \to E^{21} \to E^{13} \to 0$$

in $\mathrm{Coh}(X)$. Since $E^{13} \in \mathrm{Coh}^0(X)$ and E is locally free, $\mathrm{Ext}^1_{\mathrm{X}}(E^{13}, (-1)^*E) = 0$. Therefore, $E^{21} \cong (-1)^*E \oplus E^{13}$. Since $E^{21} \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$, we have $E^{13} = 0$ and so $E \in V^{\Phi}_{\mathrm{Coh}(X)}(2)$. Therefore, $\mathrm{ch}(E^2) = (0,0,0,-r)$. But it is not possible to have $-r > 0$ and this is the required contradiction to complete the proof. $\square$

Then we show the following.

Lemma 7.23. *Let $a, b \in \mathbb{Z}$ such that $a > 0$ with $\gcd(a,b) = 1$, and let E be a slope stable torsion free sheaf with $\mathrm{ch}_k^{b\ell/a}(E) = 0$ for $k = 1,2$. Then E^{**} is a slope stable semi-homogeneous bundle with $\mathrm{ch}(E^{**}) = (a^3, a^2b, ab^2, b^3)$.*

Proof. The slope stable torsion free sheaf E fits into the structure sequence $0 \to E \to E^{**} \to T \to 0$ for some $T \in \mathrm{Coh}^{\leq 1}(X)$. Now E^{**} is also slope stable and so by the usual Bogomolov-Gieseker inequality $\mathrm{ch}_k^{b\ell/a}(E^{**}) = 0$ for $k = 1,2$. By Theorem 7.22, $\mathcal{E}nd(E^{**})$ is a homogeneous bundle. Therefore, E^{**} is a slope stable semi-homogeneous bundle (see Lemma 2.41), and so it is a restriction of a universal bundle which is a kernel of some FM transform. Since X is principally polarized its Chern character is (a^3, a^2b, ab^2, b^3) as required. $\square$

Chapter 8

General FM Transform on Abelian Threefolds

We shall continue the setting introduced in Chapter 7 for the principally polarized abelian threefold (X, L) with Picard rank one. The aim of this chapter is to generalize the results in Chapter 7 and also to obtain further properties for any FM transform on X.

For $E \in \mathrm{Coh}(X)$ and $q \in \mathbb{Q}$, define the twisted slope

$$\mu_q(E) = \mu_{\frac{\ell}{\sqrt{6}}, q\ell}(E).$$

If $\mathrm{ch}(E) = (a_0, a_1, a_2, a_3)$ then $\mu_q(E) = (a_1/a_0) - q$ when $a_0 \neq 0$, and $\mu_q(E) = +\infty$ when $a_0 = 0$. In the rest of this thesis we mostly use μ_q slope for coherent sheaves and we simply write

$$\mathrm{HN}_q = \mathrm{HN}^{\mu}_{\frac{\ell}{\sqrt{6}}, q\ell}.$$

We also define $\mathcal{T}_q = \mathrm{HN}_q((0, +\infty])$ and $\mathcal{F}_q = \mathrm{HN}_q((-\infty, 0])$.

Let Υ and $\widehat{\Upsilon}$ be the FM transforms as introduced in Section 5.2 of Chapter 5. The isomorphisms $\widehat{\Upsilon} \circ \Upsilon \cong [-3]$ and $\Upsilon \circ \widehat{\Upsilon} \cong [-3]$ give us the following convergence of spectral sequences.

Mukai Spectral Sequence 8.1.

$$E_2^{p,q} = \widehat{\Upsilon}^p_{\mathrm{Coh}(X)} \Upsilon^q_{\mathrm{Coh}(X)}(E) \Longrightarrow H^{p+q-3}_{\mathrm{Coh}(X)}(E),$$
$$E_2^{p,q} = \Upsilon^p_{\mathrm{Coh}(X)} \widehat{\Upsilon}^q_{\mathrm{Coh}(X)}(E) \Longrightarrow H^{p+q-3}_{\mathrm{Coh}(X)}(E).$$

Let $\mathbf{R}\,\Delta$ denote the derived dualizing functor $\mathbf{R}\,\mathcal{H}om(-,\mathcal{O}_X)[3]$. Let $\widetilde{\Upsilon}$ be the FM transform with kernel the universal bundle $\mathcal{E}^*$ on $X \times X$. As in (7.1) of Chapter 7 we have the following isomorphism:

Proposition 8.2. ([PP, Lemma 2.2])

$$\mathbf{R}\,\Delta \circ \widetilde{\Upsilon} \cong (\Upsilon \circ \mathbf{R}\,\Delta)[3].$$

This gives us the convergence of the following spectral sequence.

“Duality” Spectral Sequence 8.3.

$$\Upsilon^p_{\mathrm{Coh}(X)}\left(\mathcal{E}xt^{q+3}(E,\mathcal{O}_X)\right) \Longrightarrow \; ? \quad \Longleftarrow \mathcal{E}xt^{p+3}\left(\widetilde{\Upsilon}^{3-q}_{\mathrm{Coh}(X)}(E),\mathcal{O}_X\right)$$

for $E \in \mathrm{Coh}(X)$.

Note 8.4. If $\mathrm{ch}^{q\ell}(E) = (a_0, a_1, a_2, a_3)$ then we have

$$\mathrm{ch}^{-q\ell}(\mathbf{R}\,\mathcal{H}om(E,\mathcal{O}_X)) = (a_0, -a_1, a_2, -a_3).$$

Therefore, for the FM transform $\widetilde{\Upsilon}$ we have

$$\mathrm{ch}^{w\ell/y}(\widetilde{\Upsilon}(\mathcal{O}_s)) = \mathrm{ch}^{w\ell/y}(\mathcal{E}^*_{\{s\}\times X}) = (-y^3, 0, 0, 0).$$

So the induced transform is $\widetilde{\Upsilon}^{\mathrm{H}} = \rho \begin{pmatrix} -x & y \\ z & -w \end{pmatrix}$. Similar results for abelian surfaces have been considered in [YY, Lemma 6.18].

The following proposition generalizes a series of results in Chapter 7.

Proposition 8.5. *We have the following:*

(1) *for $E \in \mathrm{Coh}(X)$*

- (i) $\Upsilon^0_{\mathrm{Coh}(X)}(E)$ *is a reflexive sheaf,*
- (ii) $\Upsilon^3_{\mathrm{Coh}(X)}(E) \in \mathcal{T}_{-w/y}$*,*
- (iii) $\Upsilon^0_{\mathrm{Coh}(X)}(E) \in \mathcal{F}_{-w/y}$*;*

(2) *for $E \in \mathcal{T}_{x/y}$*

- (i) $\Upsilon^3_{\mathrm{Coh}(X)}(E) = 0$*,*

(ii) *if* $E \in \mathrm{Coh}^{\leq 1}(X)$ *then* $\Upsilon^1_{\mathrm{Coh}(X)}(E) \in \mathcal{T}_{-w/y}$,

(iii) $\Upsilon^2_{\mathrm{Coh}(X)}(E) \in \mathcal{T}_{-w/y}$;

(3) *for* $E \in \mathcal{F}_{x/y}$

(i) $\Upsilon^0_{\mathrm{Coh}(X)}(E) = 0$,

(ii) $\Upsilon^1_{\mathrm{Coh}(X)}(E)$ *is a reflexive sheaf,*

(iii) $\Upsilon^1_{\mathrm{Coh}(X)}(E) \in \mathcal{F}_{-w/y}$.

Proof. Proofs of (1), (2) and (3) are identical to the corresponding propositions in Chapter 7 as listed below after replacing the Chern characters with their twisted counterparts.

(1) (i) Proposition 7.7, (ii) and (iii) Proposition 7.9.

(2) (i) Proposition 7.8, (ii) Proposition 7.13, (iii) Corollary 7.21.

(3) (i) Proposition 7.8, (ii) Proposition 7.10, (iii) Proposition 7.20.

□

Definition 8.6. For $\lambda \in \mathbb{Q}$, let $\mathscr{H}_\lambda$ be the abelian subcategory of $\mathrm{Coh}(X)$ generated by stable semi-homogeneous bundles having the Chern character (a^3, ba^2, b^2a, b^3) satisfying $\lambda = b/a$ and $\gcd(a,b) = 1$. Then $\mathscr{H}_0$ consists of all homogeneous bundles on X.

Let $H_\lambda \in \mathscr{H}_\lambda$. The functor $H_\lambda \otimes (-)$ is of Fourier-Mukai type with kernel $\delta_*(H_\lambda)$ on $X \times X$, where $\delta : X \to X \times X$ is the diagonal embedding. We abuse notation to write H_λ for the functor $H_\lambda \otimes (-)$. If the rank of H_λ is r then the functor H_λ induces a linear map on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ and in matrix form it is given by

$$H_\lambda^{\mathrm{H}} = r\ \rho \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}.$$

Definition 8.7. For some integer $n \geq 0$, let Φ_j, $j = 1, \dots, n+1$ be some FM transforms in the subgroup $(X \times \widehat{X}) \rtimes \widetilde{\mathrm{SL}(2,\mathbb{Z})}$ of $\mathrm{Aut}\, D^b(X)$. For $\lambda_i \in \mathbb{Q}$, $i = 1, \dots, n$ let $H_{\lambda_i} \in \mathscr{H}_{\lambda_i}$. The functor $\Pi : D^b(X) \to D^b(X)$ is defined by

$$\Pi = \Phi_{n+1} \circ H_{\lambda_n} \circ \Phi_n \circ \cdots \circ H_{\lambda_1} \circ \Phi_1[p]. \tag{8.1}$$

Image of any skyscraper sheaf $\mathcal{O}_s$ under the composition of FM functors $H_{\lambda_i} \otimes (-)$ and FM transforms in $\operatorname{Aut} D^b(X)$ is atomic with respect to $\operatorname{Coh}(X)$. In (8.1), p is some integer such that $\Pi^i_{\operatorname{Coh}(X)}(\mathcal{O}_s) = 0$ for $i \neq 0$.

Therefore, Π is an FM functor with kernel a sheaf $\mathcal{U}$ on $X \times X$. Hence, for any $E \in \operatorname{Coh}(X)$, $\Pi(E)$ can have non-trivial $\operatorname{Coh}(X)$ cohomology at $0, 1, 2, 3$ positions only. Also one can show that Π induces a linear map Π^{H} on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ given by

$$\Pi^{\mathrm{H}} = a\ \rho \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

for some $a \in \mathbb{Z}_{>0}$ and $x, y, z, w \in \mathbb{Q}$ with $xw - yz = 1$. So $\mathcal{U}_{\{s\} \times X} = \Pi(\mathcal{O}_s)$ has the Chern character $a(-y^3, y^2 w, -yw^3, w^3)$. Assume $\mathcal{U}_{\{s\} \times X}$ is not a torsion sheaf; that is $y < 0$.

Define a functor $\widehat{\Pi} : D^b(X) \to D^b(X)$ by

$$\widehat{\Pi} = (\Phi_1)^{-1} \circ H^*_{\lambda_1} \circ (\Phi_2)^{-1} \circ \cdots \circ H^*_{\lambda_n} \circ (\Phi_{n+1})^{-1} [-p-3].$$

One can check that $\widehat{\Pi}$ is an FM functor with kernel $\mathsf{Swap}^*(\mathbf{R}\,\mathcal{H}om(\mathcal{U}, \mathcal{O}_{X \times X}))$ on $X \times X$. Moreover, $\widehat{\Pi}(\mathcal{O}_s) \in \operatorname{Coh}(X)$ for any $s \in X$. So the FM kernel of $\widehat{\Pi}$ is $\mathsf{Swap}^*(\mathcal{U}^*)$. Also $\mathcal{U}$ is locally free as $\mathcal{U}_{\{s\} \times X}$ and $\mathcal{U}^*_{X \times \{s\}}$ are locally free. Moreover, for any $E \in \operatorname{Coh}(X)$, $\widehat{\Pi}(E)$ can have non-trivial $\operatorname{Coh}(X)$ cohomology at $0, 1, 2, 3$ positions only, and $\widehat{\Pi}[3]$ is left and right adjoint to Π (and vice versa). The FM functor $\widehat{\Pi}$ induces a linear map $\widehat{\Pi}^{\mathrm{H}}$ on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ given by

$$\widehat{\Pi}^{\mathrm{H}} = a\ \rho \begin{pmatrix} -w & y \\ z & -x \end{pmatrix}.$$

We have the isomorphisms

$$\widehat{\Pi} \circ \Pi \cong H_0[-3], \quad \text{and} \quad \Pi \circ \widehat{\Pi} \cong H_0[-3]$$

for some homogeneous bundle $H_0 \in \mathscr{H}_0$ with $\mathcal{O}_X$ is a direct summand of H_0. Therefore, we have the convergence of spectral sequences

$$\left.\begin{aligned} E_2^{p,q} = \widehat{\Pi}^p_{\operatorname{Coh}(X)} \Pi^q_{\operatorname{Coh}(X)}(E) \Longrightarrow H^{p+q-3}_{\operatorname{Coh}(X)}(H_0 E) \\ E_2^{p,q} = \Pi^p_{\operatorname{Coh}(X)} \widehat{\Pi}^q_{\operatorname{Coh}(X)}(E) \Longrightarrow H^{p+q-3}_{\operatorname{Coh}(X)}(H_0 E) \end{aligned}\right\} \tag{8.2}$$

for E.

Since $\mathcal{O}_X$ is a direct summand of H_0, for any $E, F \in D^b(X)$ we have

$$\mathrm{Hom}_{\mathrm{X}}(E,F) \hookrightarrow \mathrm{Hom}_{\mathrm{X}}(E, H_0 F).$$

From the adjointness $\Pi \dashv \widehat{\Pi}[3]$, $\mathrm{Hom}_{\mathrm{X}}(E, \widehat{\Pi} \circ \Pi(F)[3]) \cong \mathrm{Hom}_{\mathrm{X}}(\Pi(E), \Pi(F))$. Therefore, we have

$$\mathrm{Hom}_{\mathrm{X}}(E,F) \hookrightarrow \mathrm{Hom}_{\mathrm{X}}(\Pi(E), \Pi(F)). \tag{8.3}$$

If $\mathbf{R}\,\Delta$ denotes the derived dualizing functor $\mathbf{R}\,\mathcal{H}om(-, \mathcal{O}_X)[3]$ then $\mathbf{R}\,\Delta \circ H_\lambda^* \cong H_\lambda \circ \mathbf{R}\,\Delta$. Therefore, by iteratively using Proposition 8.2 for each of the FM transforms Φ_j together with the above isomorphism, we have

$$\mathbf{R}\,\Delta \circ \widetilde{\Pi} \cong (\Pi \circ \mathbf{R}\,\Delta)[3], \tag{8.4}$$

for some FM functor $\widetilde{\Pi}$ which is of the form (8.1). Moreover, the FM kernel of $\widetilde{\Pi}$ is $\mathcal{U}^*$ on $X \times X$ and the induced linear map on $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ of $\widetilde{\Pi}$ is

$$\widetilde{\Pi}^{\mathrm{H}} = a\ \rho \begin{pmatrix} -x & y \\ z & -w \end{pmatrix}.$$

The isomorphism (8.4) gives us the convergence of the spectral sequence

$$\Pi^p_{\mathrm{Coh}(X)} \left(\mathcal{E}xt^{q+3}(E, \mathcal{O}_X)\right) \Longrightarrow \ ? \quad \Longleftarrow \mathcal{E}xt^{p+3} \left(\widetilde{\Pi}^{3-q}_{\mathrm{Coh}(X)}(E), \mathcal{O}_X\right), \tag{8.5}$$

for $E \in \mathrm{Coh}(X)$.

The following proposition generalizes the results on FM transforms in Proposition 8.5 for FM functors of the form (8.1).

Proposition 8.8. *We have the following:*

(1) for $E \in \mathrm{Coh}(X)$

(i) $\Pi^0_{\mathrm{Coh}(X)}(E)$ *is a reflexive sheaf,*

(ii) $\Pi^3_{\mathrm{Coh}(X)}(E) \in \mathcal{T}_{-w/y}$,

(iii) $\Pi^0_{\mathrm{Coh}(X)}(E) \in \mathcal{F}_{-w/y}$;

(2) for $E \in \mathcal{T}_{x/y}$

(i) $\Pi^3_{\mathrm{Coh}(X)}(E) = 0$,

(ii) if $E \in \mathrm{Coh}^{\leq 1}(X)$ *then* $\Pi^1_{\mathrm{Coh}(X)}(E) \in \mathcal{T}_{-w/y}$,

(iii) $\Pi^2_{\mathrm{Coh}(X)}(E) \in \mathcal{T}_{-w/y}$;

(3) for $E \in \mathcal{F}_{x/y}$

(i) $\Pi^0_{\mathrm{Coh}(X)}(E) = 0$,

(ii) $\Pi^1_{\mathrm{Coh}(X)}(E)$ *is a reflexive sheaf,*

(iii) $\Pi^1_{\mathrm{Coh}(X)}(E) \in \mathcal{F}_{-w/y}$.

Proof. The proofs are similar to that of Proposition 8.5 or the series of similar results in Chapter 7. To illustrate these we shall give the proof for (1)(i) as follows.

Let $s \in X$. Then for $0 \leq i \leq 2$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, \Pi^0_{\mathrm{Coh}(X)}(E)[i]) &\hookrightarrow \mathrm{Hom}_{\mathrm{X}}(\widehat{\Pi}(\mathcal{O}_s), \widehat{\Pi}\,\Pi^0_{\mathrm{Coh}(X)}(E)[i]), && \text{by (8.3)} \\ &\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{U}^*_{X\times\{s\}}, \widehat{\Pi}^2_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}[-2+i]) \end{aligned}$$

from the convergence of the Spectral Sequence (8.2) for E. So $\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, \Pi^0_{\mathrm{Coh}(X)}(E))$ $= \mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_s, \Pi^0_{\mathrm{Coh}(X)}(E)) = 0$, and

$$\begin{aligned} \mathrm{Ext}^2_{\mathrm{X}}(\mathcal{O}_s, \Pi^0_{\mathrm{Coh}(X)}(E)) &\hookrightarrow \mathrm{Hom}_{\mathrm{X}}(\mathcal{U}^*_{X\times\{s\}}, \widehat{\Pi}^2_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E)) \\ &\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{U}^*_{X\times\{s\}}, \widehat{\Pi}^0_{\mathrm{Coh}(X)}\Pi^1_{\mathrm{Coh}(X)}(E)), \quad \text{by Spec. Seq. (8.2)} \\ &\cong \mathrm{Hom}_{\mathrm{X}}(\widehat{\Pi}(\mathcal{O}_s), \widehat{\Pi}\,\Pi^1_{\mathrm{Coh}(X)}(E)) \\ &\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, \Pi\,\widehat{\Pi}\,\Pi^1_{\mathrm{Coh}(X)}(E)[3]), && \text{by } \widehat{\Pi} \dashv \Pi[3] \\ &\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, H_0\Pi^1_{\mathrm{Coh}(X)}(E)). \end{aligned}$$

Hence, $\dim \mathrm{Sing}\left(\Pi^0_{\mathrm{Coh}(X)}(E)\right) = \dim\{s \in X : \mathrm{Ext}^2_{\mathrm{X}}(\mathcal{O}_s, \Pi^0_{\mathrm{Coh}(X)}(E)) \neq 0\} \leq 0$. Therefore, $\Pi^0_{\mathrm{Coh}(X)}(E)$ is a reflexive sheaf (from Lemma 2.47). □

Proposition 8.9. *For* $\lambda \in \mathbb{Q}_{>0}$,

(i) if $E \in \mathrm{HN}_{x/y}((0, \lambda])$ *then* $\Pi^0_{\mathrm{Coh}(X)}(E) \in \mathrm{HN}_{-w/y}((-\infty, -\frac{1}{2\lambda y^2}])$,

(ii) if $E \in \mathrm{HN}_{x/y}([-\lambda, 0])$ *then* $\Pi^3_{\mathrm{Coh}(X)}(E) \in \mathrm{HN}_{-w/y}([\frac{1}{2\lambda y^2}, +\infty])$.

Proof. (i) Let $E \in \mathrm{HN}_{x/y}((0,\lambda])$. Pick a bundle $H_{-\lambda} \in \mathscr{H}_{-\lambda}$ of rank r. Let Ξ be the FM functor defined by

$$\Xi = \Pi \circ H_{-\lambda} \circ \widehat{\Pi}[3].$$

The induced liner map of Ξ on $H^{2*}_{\mathrm{alg}}(X,\mathbb{Q})$ is

$$\Xi^{\mathrm{H}} = -ra^2\ \rho \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} -w & y \\ z & -x \end{pmatrix} = ra^2\ \rho \begin{pmatrix} 1+\lambda yw & -\lambda y^2 \\ \lambda w^2 & 1-\lambda yw \end{pmatrix}.$$

The isomorphism $\Xi \circ \Pi \cong \Pi \circ H_{-\lambda} \circ H_0$ gives us the convergence of spectral sequence:

$$E_2^{p,q} = \Xi^p_{\mathrm{Coh}(X)} \Pi^q_{\mathrm{Coh}(X)}(E) \Longrightarrow \Pi^{p+q}_{\mathrm{Coh}(X)}(H'_{-\lambda}E)$$

for E. Here $H'_{-\lambda} = H_{-\lambda}H_0$. So $H'_{-\lambda}E \in \mathrm{HN}_{x/y}((-\lambda,0])$. By (3)(i) of Proposition 8.8, $\Pi^0_{\mathrm{Coh}(X)}(H'_{-\lambda}E) = 0$. Now from the convergence of the above spectral sequence, $\Xi^0_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E) = 0$ and $\Xi^1_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E) \hookrightarrow \Pi^1_{\mathrm{Coh}(X)}(H'_{-\lambda}E)$. By (3)(iii) of Proposition 8.8, $\Pi^1_{\mathrm{Coh}(X)}(H'_{-\lambda}E) \in \mathrm{HN}_{-w/y}((-\infty,0])$. Since we have $\mathrm{HN}_{-w/y}((-\infty,0]) \subset \mathrm{HN}_{(1-\lambda yw)/\lambda y^2}((-\infty,0])$,

$$\Xi^1_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E) \in \mathrm{HN}_{(1-\lambda yw)/\lambda y^2}((-\infty,0]). \tag{8.6}$$

By the Harder-Narasimhan filtration, $\Pi^0_{\mathrm{Coh}(X)}(E) \in \mathrm{HN}_{-w/y}((-\infty,0])$ fits into the short exact sequence

$$0 \to F \to \Pi^0_{\mathrm{Coh}(X)}(E) \to G \to 0 \tag{8.7}$$

in $\mathrm{Coh}(X)$ for some $F \in \mathrm{HN}_{-w/y}((-\frac{1}{2\lambda y^2},0])$ and $G \in \mathrm{HN}_{-w/y}((-\infty,-\frac{1}{2\lambda y^2}])$. Assume $F \neq 0$ for a contradiction. Then we can write $\mathrm{ch}^{-w\ell/y}(F) = (a_0,\mu a_0,a_2,a_3)$ for $0 \geq \mu > -\frac{1}{2\lambda y^2}$.

By applying the FM functor $\widehat{\Pi}$ to short exact sequence (8.7) we have the following exact sequence in $\mathrm{Coh}(X)$:

$$0 \to \widehat{\Pi}^1_{\mathrm{Coh}(X)}(G) \to \widehat{\Pi}^2_{\mathrm{Coh}(X)}(F) \to \widehat{\Pi}^2_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E) \to \cdots .$$

By Spectral Sequence (8.2), $\widehat{\Pi}^2_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E) \cong \widehat{\Pi}^0_{\mathrm{Coh}(X)}\Pi^1_{\mathrm{Coh}(X)}(E)$ and so by (1)(iii) of Proposition 8.8, it is in $\mathrm{HN}_{x/y}((-\infty,0])$. Also by (3)(iii) of Proposition 8.8, $\widehat{\Pi}^1_{\mathrm{Coh}(X)}(G) \in \mathrm{HN}_{x/y}((-\infty,0])$. Therefore, $\widehat{\Pi}^2_{\mathrm{Coh}(X)}(F) \in \mathrm{HN}_{x/y}((-\infty,0])$.

By (1)(ii) of Proposition 8.8, $\widehat{\Pi}^3_{\mathrm{Coh}(X)}(F) \in \mathrm{HN}_{x/y}((0,+\infty])$. Therefore, we have $\ell^2 \operatorname{ch}_1^{x\ell/y}(\widehat{\Pi}(F)) \leq 0$, and so $ya_2 \leq 0$.

Since $\Xi^0_{\mathrm{Coh}(X)}(F) \hookrightarrow \Xi^0_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E)$ we have $\Xi^0_{\mathrm{Coh}(X)}(F) = 0$. Moreover, since $F \in \mathrm{HN}_{-w/y}((-\frac{1}{2\lambda y^2}, 0]) = \mathrm{HN}_{-(1+\lambda yw)/\lambda y^2}((\frac{1}{2\lambda y^2}, \frac{1}{\lambda y^2}])$ we have $\Xi^3_{\mathrm{Coh}(X)}(F) = 0$. Apply the FM functor Ξ to short exact sequence (8.7) and consider the long exact sequence of $\mathrm{Coh}(X)$-cohomologies:

$$0 \to \Xi^0_{\mathrm{Coh}(X)}(G) \to \Xi^1_{\mathrm{Coh}(X)}(F) \to \Xi^1_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E) \to \cdots .$$

By (8.6), $\Xi^1_{\mathrm{Coh}(X)}\Pi^0_{\mathrm{Coh}(X)}(E) \in \mathrm{HN}_{(1-\lambda yw)/\lambda y^2}((-\infty, 0])$, and by (1)(iii) of Proposition 8.8, $\Xi^0_{\mathrm{Coh}(X)}(G) \in \mathrm{HN}_{(1-\lambda yw)/\lambda y^2}((-\infty, 0])$.

Therefore, $\Xi^1_{\mathrm{Coh}(X)}(F) \in \mathrm{HN}_{(1-\lambda yw)/\lambda y^2}((-\infty, 0])$. By (2)(iii) of Proposition 8.8, $\Xi^2_{\mathrm{Coh}(X)}(F) \in \mathrm{HN}_{(1-\lambda yw)/\lambda y^2}((0, +\infty])$. So $\ell^2 \operatorname{ch}_1^{(1-\lambda yw)\ell/\lambda y^2}(\Xi(F)) \geq 0$.

On the other hand, we have

$$\begin{aligned}
&\operatorname{ch}^{(1-\lambda yw)\ell/\lambda y^2}(\Xi(F)) \\
&\qquad = ra^2\, \rho \begin{pmatrix} 1 & 0 \\ \frac{1-\lambda yw}{\lambda y^2} & 1 \end{pmatrix} \begin{pmatrix} 1+\lambda yw & -\lambda y^2 \\ \lambda w^2 & 1-\lambda yw \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{w}{y} & 1 \end{pmatrix} \operatorname{ch}^{-w\ell/y}(F) \\
&\qquad = ra^2\, \rho \begin{pmatrix} 1 & -\lambda y^2 \\ \frac{1}{\lambda y^2} & 0 \end{pmatrix} \operatorname{ch}^{-w\ell/y}(F) \\
&\qquad = ra^2 \begin{pmatrix} * & * & * & * \\ -\frac{1}{\lambda y^2} & -2 & -\lambda y^2 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \begin{pmatrix} a_0 \\ \mu a_0 \\ a_2 \\ a_3 \end{pmatrix}, \quad \text{by Proposition 4.5} \\
&\qquad = ra^2 \left(*, -2a_0 \left(\mu + \frac{1}{2\lambda y^2} \right) - \lambda^2 y^2 a_2, *, * \right).
\end{aligned}$$

Here $a_0 > 0$, $\left(\mu + \frac{1}{2\lambda y^2}\right) > 0$, $ya_2 \leq 0$ and so $\ell^2 \operatorname{ch}_1^{(1-\lambda yw)\ell/\lambda y^2}(\Xi(F)) < 0$. This is the required contradiction.

(ii) Let $E \in \mathrm{HN}_{x/y}([-\lambda, 0])$ for some $\lambda \in \mathbb{Q}_{>0}$. From Spectral Sequence (8.5) for E we have

$$\left(\Pi^3_{\mathrm{Coh}(X)}(E)\right)^* \cong \widetilde{\Pi}^0_{\mathrm{Coh}(X)}(E^*).$$

Here $\widetilde{\Pi}^{\mathrm{H}} = a\ \rho \begin{pmatrix} -x & y \\ z & -w \end{pmatrix}$ and we have $E^* \in \mathrm{HN}_{-x/y}([0,\lambda])$. So by (3)(i) of Proposition 8.8 and the above result, we have $\widetilde{\Pi}^0_{\mathrm{Coh}(X)}(E^*) \in \mathrm{HN}_{w/y}((-\infty, -\frac{1}{2\lambda y^2}])$. Therefore, $\left(\Pi^3_{\mathrm{Coh}(X)}(E)\right)^* \in \mathrm{HN}_{w/y}((-\infty, -\frac{1}{2\lambda y^2}])$, and so we have $\Pi^3_{\mathrm{Coh}(X)}(E) \in \mathrm{HN}_{-w/y}([\frac{1}{2\lambda y^2}, +\infty])$ as required.

□

Recall from (5.1) in Chapter 5, for $\lambda \in \mathbb{Q}_{>0}$, $b = \left(\frac{x}{y} + \frac{\lambda}{2}\right)$, $m = \frac{\sqrt{3}\lambda}{2}$, $b' = \left(-\frac{w}{y} - \frac{1}{2\lambda y^2}\right)$ and $m' = \frac{\sqrt{3}}{2\lambda y^2}$. Let $\Upsilon, \widehat{\Upsilon}$ be the FM transforms as introduced in Section 5.2 of Chapter 5.

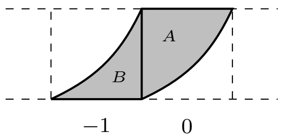
Theorem 8.10. *We have the following:*

(i) $\Upsilon\left(\mathcal{B}_{m\ell,b\ell}\right) \subset \langle \mathcal{B}_{m'\ell,b'\ell}, \mathcal{B}_{m'\ell,b'\ell}[-1], \mathcal{B}_{m'\ell,b'\ell}[-2]\rangle$, *and*

(ii) $\widehat{\Upsilon}[1]\left(\mathcal{B}_{m'\ell,b'\ell}\right) \subset \langle \mathcal{B}_{m\ell,b\ell}, \mathcal{B}_{m\ell,b\ell}[-1], \mathcal{B}_{m\ell,b\ell}[-2]\rangle$.

Proof. (i) We can visualize $\mathcal{B}_{m\ell,b\ell}$ and $\mathcal{B}_{m'\ell,b'\ell}$ as follows:

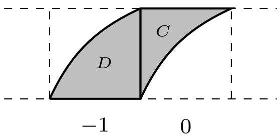
$\mathcal{B}_{m\ell,b\ell} = \langle \mathcal{F}_b[1], \mathcal{T}_b\rangle$:



$A \in \mathcal{T}_b = \mathrm{HN}_{x/y}((\frac{\lambda}{2}, +\infty])$

$B \in \mathcal{F}_b = \mathrm{HN}_{x/y}((-\infty, \frac{\lambda}{2}])$

$\mathcal{B}_{m'\ell,b'\ell} = \langle \mathcal{F}_{b'}[1], \mathcal{T}_{b'}\rangle$:

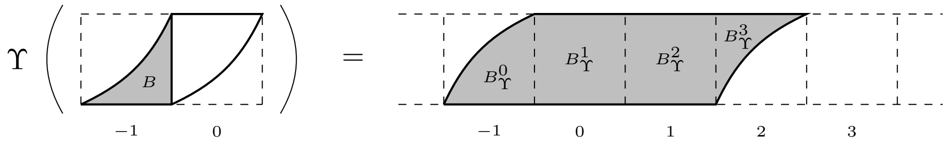


$C \in \mathcal{T}_{b'} = \mathrm{HN}_{-w/y}((-\frac{1}{2\lambda y^2}, +\infty])$

$D \in \mathcal{F}_{b'} = \mathrm{HN}_{-w/y}((-\infty, -\frac{1}{2\lambda y^2}])$

If $E \in \mathcal{F}_b = \mathrm{HN}_{x/y}((-\infty, \frac{\lambda}{2}])$ then by (3)(i) of Proposition 8.5 and (i) of Proposition 8.9, $\Upsilon^0_{\mathrm{Coh}(X)}(E) \in \mathcal{F}_{b'}$. Also by (1)(ii) of Proposition 8.5, $\Upsilon^3_{\mathrm{Coh}(X)}(E) \in \mathcal{T}_{-w/y} \subset \mathcal{T}_{b'}$. Therefore, $\Upsilon(E)$ has $\mathcal{B}_{m'\ell,b'\ell}$-cohomologies in 1,2,3 positions. That is

$$\Upsilon\left(\mathcal{F}_b\right)[1] \subset \langle \mathcal{B}_{m'\ell,b'\ell}, \mathcal{B}_{m'\ell,b'\ell}[-1], \mathcal{B}_{m'\ell,b'\ell}[-2]\rangle.$$

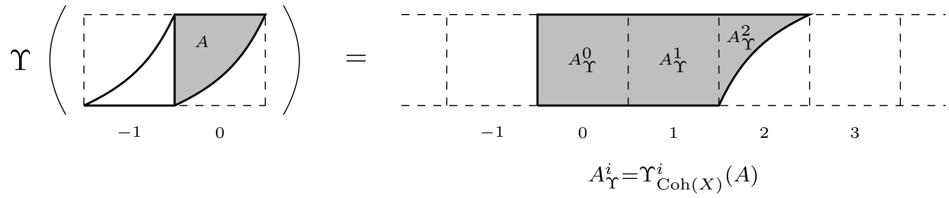


$B^i_\Upsilon = \Upsilon^i_{\mathrm{Coh}(X)}(B)$

On the other hand, if $E \in \mathcal{T}_b = \mathrm{HN}_{x/y}((\frac{\lambda}{2}, +\infty])$ then by (2)(i) of Proposition 8.5, $\Upsilon^3_{\mathrm{Coh}(X)}(E) = 0$, and by (2)(iii) of Proposition 8.5, $\Upsilon^2_{\mathrm{Coh}(X)}(E) \in$

$\mathrm{HN}_{-w/y}((0,+\infty]) \subset \mathcal{T}_{b'}$. So $\Upsilon(E)$ has $\mathcal{B}_{m'\ell,b'\ell}$-cohomologies in positions 0,1,2 only. That is

$$\Upsilon\left(\mathcal{T}_b\right) \subset \langle \mathcal{B}_{m'\ell,b'\ell}, \mathcal{B}_{m'\ell,b'\ell}[-1], \mathcal{B}_{m'\ell,b'\ell}[-2] \rangle.$$



Hence, $\Upsilon\left(\mathcal{B}_{m\ell,b\ell}\right) \subset \langle \mathcal{B}_{m'\ell,b'\ell}, \mathcal{B}_{m'\ell,b'\ell}[-1], \mathcal{B}_{m'\ell,b'\ell}[-2] \rangle$, as $\mathcal{B}_{m\ell,b\ell} = \langle \mathcal{F}_b[1], \mathcal{T}_b \rangle$.

(ii) We can use (3)(i), (3)(iii), (2)(i), (1)(iii) of Proposition 8.5 and (ii) of Proposition 8.9 in a similar way to the above proof.

$\square$

Chapter 9

Equivalences of Abelian Threefold Hearts

The aim of this chapter is to complete the proof of Theorem 5.5 in Chapter 5.

It will be convenient to abbreviate the FM transforms Υ and $\widehat{\Upsilon}[1]$ by Γ and $\widehat{\Gamma}$ respectively. Then by Theorem 8.10, the images of an object from $\mathcal{B}_{m\ell,b\ell}$ (and $\mathcal{B}_{m'\ell,b'\ell}$) under Γ (and $\widehat{\Gamma}$) are complexes whose cohomologies with respect to $\mathcal{B}_{m'\ell,b'\ell}$ (and $\mathcal{B}_{m\ell,b\ell}$) can only be non-zero in $0, 1, 2$ positions.

The abelian category $\mathcal{B}_{m\ell,b\ell} = \langle \mathcal{F}_b[1], \mathcal{T}_b \rangle$ does not depend on $m > 0$. So in the rest of the thesis we write

$$\Gamma^i_b(E) = H^i_{\mathcal{B}_{m\ell,b\ell}}(\Gamma(E)).$$

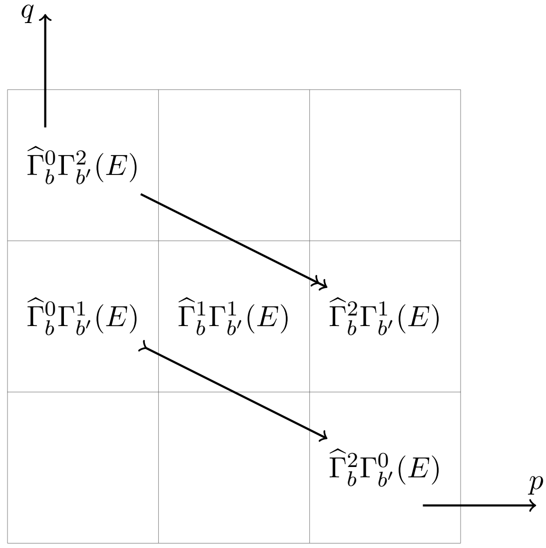
We have $\Gamma \circ \widehat{\Gamma} \cong [-2]$ and $\widehat{\Gamma} \circ \Gamma \cong [-2]$. This gives us the following convergence of spectral sequences.

Spectral Sequence 9.1.

(1) $E_2^{p,q} = \widehat{\Gamma}^p_b \Gamma^q_{b'}(E) \Longrightarrow H^{p+q-2}_{\mathcal{B}_{m\ell,b\ell}}(E)$, *and*

(2) $E_2^{p,q} = \Gamma^p_{b'} \widehat{\Gamma}^q_b(E) \Longrightarrow H^{p+q-2}_{\mathcal{B}_{m'\ell,b'\ell}}(E)$.

Such convergence of the spectral sequences for $E \in \mathcal{B}_{m\ell,b\ell}$ and $E \in \mathcal{B}_{m'\ell,b'\ell}$ behave in the same way as the convergence of the Mukai Spectral Sequence 6.1 for coherent sheaves on an abelian surface. The following diagram describes the convergence of Spectral Sequence 9.1–(1) for $E \in \mathcal{B}_{m\ell,b\ell}$.



Proposition 9.2. *We have the following:*

(1) For $E \in \mathcal{T}'_{m'\ell,b'\ell}$,

(i) $H^0_{\mathrm{Coh}(X)}(\widehat{\Gamma}^2_b(E)) = 0$, and

(ii) if $\widehat{\Gamma}^2_b(E) \neq 0$ then $\Im Z_{m\ell,b\ell}(\widehat{\Gamma}^2_b(E)) > 0$.

(2) For $E \in \mathcal{F}'_{m'\ell,b'\ell}$,

(i) $H^{-1}_{\mathrm{Coh}(X)}(\widehat{\Gamma}^0_b(E)) = 0$, and

(ii) if $\widehat{\Gamma}^0_b(E) \neq 0$ then $\Im Z_{m\ell,b\ell}(\widehat{\Gamma}^0_b(E)) < 0$.

(3) For $E \in \mathcal{T}'_{m\ell,b\ell}$,

(i) $H^0_{\mathrm{Coh}(X)}(\Gamma^2_{b'}(E)) = 0$, and

(ii) if $\Gamma^2_{b'}(E) \neq 0$ then $\Im Z_{m'\ell,b'\ell}(\Gamma^2_{b'}(E)) > 0$.

(4) For $E \in \mathcal{F}'_{m\ell,b\ell}$,

(i) $H^{-1}_{\mathrm{Coh}(X)}(\Gamma^0_{b'}(E)) = 0$, and

(ii) if $\Gamma^0_{b'}(E) \neq 0$ then $\Im Z_{m'\ell,b'\ell}(\Gamma^0_{b'}(E)) < 0$.

Proof. (1) Let $E \in \mathcal{T}'_{m'\ell,b'\ell}$.

(i) For any $s \in X$,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{X}}(\widehat{\Gamma}^2_b(E), \mathcal{O}_s) &\cong \mathrm{Hom}_{\mathrm{X}}(\widehat{\Gamma}^2_b(E), \widehat{\Gamma}^2_b(\mathcal{E}_{\{s\}\times X}))\\ &\cong \mathrm{Hom}_{\mathrm{X}}(\widehat{\Gamma}(E), \widehat{\Gamma}(\mathcal{E}_{\{s\}\times X}))\end{aligned}$$

$$\cong \mathrm{Hom}_{\mathrm{X}}(E, \mathcal{E}_{\{s\}\times X}) = 0,$$

since $E \in \mathcal{T}'_{m'\ell,b'\ell}$ and $\mathcal{E}_{\{s\}\times X} \in \mathcal{F}'_{m'\ell,b'\ell}$. Therefore, $H^0_{\mathrm{Coh}(X)}(\widehat{\Gamma}^2_b(E)) = 0$ as required.

(ii) From (1)(i), we have $\widehat{\Gamma}^2_b(E) \cong A[1]$ for some $0 \neq A \in \mathrm{HN}_{x/y}((-\infty, \frac{\lambda}{2}])$. Consider the convergence of the spectral sequence:

$$E_2^{p,q} = \widehat{\Gamma}^p_{\mathrm{Coh}(X)}(H^q_{\mathrm{Coh}(X)}(E)) \Longrightarrow \widehat{\Gamma}^{p+q}_{\mathrm{Coh}(X)}(E)$$

for E. If $E_i = H^i_{\mathrm{Coh}(X)}(E)$, then by Proposition 3.16, we have $E_0 \in \mathrm{HN}_{-w/y}((0, +\infty])$ and so by (2)(iii) and (1)(ii) of Proposition 8.5,

$$\widehat{\Upsilon}^2_{\mathrm{Coh}(X)}(E_0), \widehat{\Upsilon}^3_{\mathrm{Coh}(X)}(E_{-1}) \in \mathrm{HN}_{x/y}((0, +\infty]).$$

Therefore, from the convergence of the above spectral sequence for E, we have

$$A \in \mathrm{HN}_{x/y}((-\infty, \frac{\lambda}{2}]) \cap \mathrm{HN}_{x/y}((0, +\infty]) = \mathrm{HN}_{x/y}((0, \frac{\lambda}{2}]).$$

Let $\mathrm{ch}^{x\ell/y}(A) = (a_0, a_1, a_2, a_3)$. From the usual Bogomolov-Gieseker inequalities for all the Harder-Narasimhan semistable factors of A we have $\frac{\lambda}{2}a_1 - a_2 \geq 0$ and so

$$\Im Z_{m\ell,b\ell}(\widehat{\Gamma}^2_b(E)) = \Im Z_{m\ell,b\ell}(A[1]) = \frac{3\sqrt{3}\lambda}{2}(\lambda a_1 - a_2) > 0$$

as required.

(2) Let $E \in \mathcal{F}'_{m'\ell,b'\ell}$.

(i) For any $s \in X$ we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{X}}(\widehat{\Gamma}^0_b(E), \mathcal{O}_s[1]) &\cong \mathrm{Hom}_{\mathrm{X}}(\Gamma\widehat{\Gamma}^0_b(E), \Gamma(\mathcal{O}_s[1])) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(\Gamma^2_{b'}\widehat{\Gamma}^0_b(E)[-2], \mathcal{E}_{\{s\}\times X}[1]) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(\Gamma^2_{b'}\widehat{\Gamma}^0_b(E), \mathcal{E}_{\{s\}\times X}[3]) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, \Gamma^2_{b'}\widehat{\Gamma}^0_b(E))^*.
\end{aligned}$$

From the convergence of the Spectral Sequence 9.1 for E, we have the short exact sequence

$$0 \to \Gamma^0_{b'}\widehat{\Gamma}^1_b(E) \to \Gamma^2_{b'}\widehat{\Gamma}^0_b(E) \to F \to 0$$

in $\mathcal{B}_{m'\ell,b'\ell}$, where F is a subobject of E and so $F \in \mathcal{F}'_{m'\ell,b'\ell}$. Moreover, by the Harder-Narasimhan filtration, F fits into the following short exact sequence in $\mathcal{B}_{m'\ell,b'\ell}$:

$$0 \to F_0 \to F \to F_1 \to 0,$$

where $F_0 \in \mathrm{HN}^{\nu}_{m'\ell,b'\ell}(0)$ and $F_1 \in \mathrm{HN}^{\nu}_{m'\ell,b'\ell}((-\infty,0))$. Since $\mathcal{E}_{\{s\}\times X} \in \mathrm{HN}^{\nu}_{m'\ell,b'\ell}(0)$,

$$\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, F_1) = 0.$$

Moreover, F_0 fits into a filtration with quotients of $\nu_{m'\ell,b'\ell}$-stable objects $F_{0,i}$ with $\nu_{m'\ell,b'\ell}(F_{0,i}) = 0$. By Proposition 3.13, each $F_{0,i}$ fits into a non-splitting short exact sequence

$$0 \to F_{0,i} \to M_i \to T_i \to 0$$

in $\mathcal{B}_{m'\ell,b'\ell}$ for some $T_i \in \mathrm{Coh}^0(X)$ such that $M_i[1] \in \mathcal{A}_{m'\ell,b'\ell}$ is a minimal object. Moreover, $\mathcal{E}_{\{s\}\times X}[1] \in \mathcal{A}_{m'\ell,b'\ell}$ is a minimal object. So finitely many $s \in X$ we can have $\mathcal{E}_{\{s\}\times X} \cong M_i$ for some i. So for generic $s \in X$, $\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, M_i) = 0$ and so $\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, F_{0,i}) = 0$ which implies $\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, F_0) = 0$. Therefore, for generic $s \in X$, $\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, F) = 0$. On the other hand,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, \Gamma^0_{b'}\widehat{\Gamma}^1_b(E)) &\cong \mathrm{Hom}_{\mathrm{X}}(\Gamma^0_{b'}(\mathcal{O}_s), \Gamma^0_{b'}\widehat{\Gamma}^1_b(E)) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(\Gamma(\mathcal{O}_s), \Gamma\widehat{\Gamma}^1_b(E)) \\
&\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, \widehat{\Gamma}^1_b(E)).
\end{aligned}$$

Here $\widehat{\Gamma}^1_b(E)$ fits into the short exact sequence

$$0 \to H^{-1}_{\mathrm{Coh}(X)}(\widehat{\Gamma}^1_b(E))[1] \to \widehat{\Gamma}^1_b(E) \to H^0_{\mathrm{Coh}(X)}(\widehat{\Gamma}^1_b(E)) \to 0$$

in $\mathcal{B}_{m\ell,b\ell}$, where $H^{-1}_{\mathrm{Coh}(X)}(\widehat{\Gamma}^1_b(E))$ is torsion free and $H^0_{\mathrm{Coh}(X)}(\widehat{\Gamma}^1_b(E))$ can have torsion supported on a 0-subscheme of finite length. Hence, for generic $s \in X$, $\mathrm{Hom}_{\mathrm{X}}(\mathcal{O}_s, \widehat{\Gamma}^1_b(E)) = 0$. Therefore, for generic $s \in X$, we have

$\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, \Gamma^0_{b'}\widehat{\Gamma}^1_b(E)) = \mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, F) = 0.$

So $\mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, \Gamma^2_{b'}\widehat{\Gamma}^0_b(E)) = 0$. Hence, for generic $s \in X$

$$\mathrm{Hom}_{\mathrm{X}}(\widehat{\Gamma}^0_b(E), \mathcal{O}_s[1]) = 0.$$

But $H^{-1}_{\mathrm{Coh}(X)}(\widehat{\Gamma}^0_b(E))$ is torsion free and so $H^{-1}_{\mathrm{Coh}(X)}(\widehat{\Gamma}^0_b(E)) = 0$ as required.

(ii) From (2)(i) we have $\widehat{\Gamma}^0_b(E) \cong A$ for some non-trivial coherent sheaf $A \in \mathrm{HN}_{x/y}((\frac{\lambda}{2}, +\infty])$. For any $s \in X$ we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_s, A) &\cong \mathrm{Ext}^1_{\mathrm{X}}(\mathcal{O}_s, \widehat{\Gamma}^0_b(E)) \cong \mathrm{Hom}_{\mathrm{X}}(\Gamma(\mathcal{O}_s), \Gamma\widehat{\Gamma}^0_b(E)[1]) \\ &\cong \mathrm{Hom}_{\mathrm{X}}(\mathcal{E}_{\{s\}\times X}, \Gamma^2_{b'}\widehat{\Gamma}^0_b(E)[-1]) = 0.\end{aligned}$$

So $A \in \mathrm{Coh}^{\geq 2}(X)$, and if $\mathrm{ch}^{x\ell/y}(A) = (a_0, a_1, a_2, a_3)$ then we have $a_1 > 0$. Apply the FM transform Γ to $\widehat{\Gamma}^0_b(E)$. Since $\widehat{\Gamma}^0_b(E) \in V^{\Gamma}_{\mathcal{B}_{m'\ell,b'\ell}}(2)$, $\Gamma^2_{b'}\widehat{\Gamma}^0_b(E) \in \mathcal{B}_{m'\ell,b'\ell}$ has $\mathrm{Coh}(X)$-cohomologies:

- $\Upsilon^1_{\mathrm{Coh}(X)}(A)$ in position -1, and
- $\Upsilon^2_{\mathrm{Coh}(X)}(A)$ in position 0.

So we have $A \in V^{\Upsilon}_{\mathrm{Coh}(X)}(1,2)$, $\Upsilon^1_{\mathrm{Coh}(X)}(A) \in \mathrm{HN}_{-w/y}((-\infty, -\frac{1}{2\lambda y^2}])$, and by (2)(iii) of Proposition 8.5, $\Upsilon^2_{\mathrm{Coh}(X)}(A) \in \mathrm{HN}_{-w/y}((0, +\infty])$. Therefore, $\ell^2 \mathrm{ch}_1^{-w\ell/y}(\Upsilon^1_{\mathrm{Coh}(X)}(A)) \leq 0$ and $\ell^2 \mathrm{ch}_1^{-w\ell/y}(\Upsilon^2_{\mathrm{Coh}(X)}(A)) \geq 0$. Hence,

$$\begin{aligned}a_2\ell^3 &= 2\ell\,\mathrm{ch}_2^{x\ell/y}(A) = \frac{1}{y}\ell^2\,\mathrm{ch}_1^{-w\ell/y}(\Upsilon(A)) \\ &= -\frac{1}{|y|}\left(-\ell^2\,\mathrm{ch}_1^{-w\ell/y}(\Upsilon^1_{\mathrm{Coh}(X)}(A)) + \ell^2\,\mathrm{ch}_1^{-w\ell/y}(\Upsilon^2_{\mathrm{Coh}(X)}(A))\right) \leq 0.\end{aligned}$$

So

$$\Im Z_{m\ell,b\ell}(\widehat{\Gamma}^0_b(E)) = \Im Z_{m\ell,b\ell}(A) = \frac{3\sqrt{3}\lambda}{2}(a_2 - \lambda a_1) < 0$$

as required.

(3) Let $E \in \mathcal{T}'_{m\ell,b\ell}$.

(i) Similar to the proof of (1)(i).

(ii) From (3)(i), we have $\Gamma^2_{b'}(E) \cong A[1]$ for some coherent sheaf $0 \neq A \in \mathrm{HN}_{-w/y}((-\infty, -\frac{1}{2\lambda y^2}])$. Let $\mathrm{ch}^{-w\ell/y}(A) = (a_0, a_1, a_2, a_3)$. So $a_1 < 0$.

Apply the FM transform $\widehat{\Gamma}$ to $\Gamma^2_{b'}(E)$. Since $\Gamma^2_{b'}(E) \in V^{\widehat{\Gamma}}_{\mathcal{B}_{m\ell,b\ell}}(0)$, $\widehat{\Gamma}^0_b\Gamma^2_{b'}(E) \in \mathcal{B}_{m\ell,b\ell}$ has $\mathrm{Coh}(X)$-cohomologies:

- $\widehat{\Upsilon}^1_{\mathrm{Coh}(X)}(A)$ in position -1, and
- $\widehat{\Upsilon}^2_{\mathrm{Coh}(X)}(A)$ in position 0.

So we have $A \in V^{\widehat{\Upsilon}}_{\mathrm{Coh}(X)}(1,2)$, $\widehat{\Upsilon}^2_{\mathrm{Coh}(X)}(A) \in \mathrm{HN}_{x/y}([\frac{\lambda}{2}, +\infty])$, and by (3)(iii) of Proposition 8.5, $\widehat{\Upsilon}^1_{\mathrm{Coh}(X)}(A) \in \mathrm{HN}_{x/y}((-\infty, 0])$.

Therefore, $\ell^2 \operatorname{ch}_1^{x\ell/y}(\widehat{\Upsilon}^1_{\mathrm{Coh}(X)}(A)) \leq 0$ and $\ell^2 \operatorname{ch}_1^{x\ell/y}(\widehat{\Upsilon}^2_{\mathrm{Coh}(X)}(A)) \geq 0$. So $ya_2\ell^3 = \ell^2 \operatorname{ch}_1^{x\ell/y}(\widehat{\Upsilon}(A)) \geq 0$. Since $y < 0$, we have

$$\Im Z_{m'\ell,b'\ell}(\Gamma^2_{b'}(E)) = \Im Z_{m'\ell,b'\ell}(A[1]) = \frac{3\sqrt{3}}{2\lambda y^2}\left(-a_2 - \frac{1}{\lambda y^2}a_1\right) > 0$$

as required.

(4) Let $E \in \mathcal{F}'_{m\ell,b\ell}$.

(i) Similar to the proof of (2)(i).

(ii) From (4)(i) we have $\Gamma^0_{b'}(E) \cong A$ for some non-trivial coherent sheaf $A \in \mathcal{T}_{b'} = \mathrm{HN}_{-w/y}((-\frac{1}{2\lambda y^2}, +\infty])$.

Consider the convergence of the spectral sequence for E:

$$E_2^{p,q} = \Gamma^p_{\mathrm{Coh}(X)}(H^q_{\mathrm{Coh}(X)}(E)) \Longrightarrow \Gamma^{p+q}_{\mathrm{Coh}(X)}(E).$$

Let $E_i = H^i_{\mathrm{Coh}(X)}(E)$. By Proposition 3.15, we have $E_{-1} \in \mathrm{HN}_{x/y}((-\infty, 0])$, and so by (3)(iii) and (1)(iii) of Proposition 8.5,

$$\Gamma^1_{\mathrm{Coh}(X)}(E_{-1}) \in \mathrm{HN}_{-w/y}((-\infty, 0]), \text{ and } \Gamma^0_{\mathrm{Coh}(X)}(E_0) \in \mathrm{HN}_{-w/y}((-\infty, 0]).$$

Therefore, from the convergence of the above spectral sequence for E, we have

$$A \in \mathrm{HN}_{-w/y}((-\frac{1}{2\lambda y^2}, +\infty]) \cap \mathrm{HN}_{-w/y}((-\infty, 0]) = \mathrm{HN}_{-w/y}((-\frac{1}{2\lambda y^2}, 0]).$$

Also by (3)(ii) and (1)(i) of Proposition 8.5, $\Gamma^1_{\mathrm{Coh}(X)}(E_{-1})$ and $\Gamma^0_{\mathrm{Coh}(X)}(E_0)$ are reflexive sheaves, and so A is reflexive. Let $\operatorname{ch}^{-w\ell/y}(A) = (a_0, a_1, a_2, a_3)$. By the usual Bogomolov-Gieseker inequalities for all the Harder-Narasimhan semistable factors of A, we obtain $a_2 + \frac{1}{2\lambda y^2} a_1 \leq 0$. So we have

$$\Im Z_{m'\ell,b'\ell}(\Gamma^0_{b'}(E)) = \Im Z_{m'\ell,b'\ell}(\Gamma^0_{b'}(A)) = \frac{3\sqrt{3}}{2\lambda y^2}\left(a_2 + \frac{1}{\lambda y^2} a_1\right) \leq 0.$$

Equality holds when $A \in \mathrm{HN}_{-w/y}(0)$ with $\mathrm{ch}^{-w\ell/y}(A) = (a_0, 0, 0, *)$. By considering a Jordan-Hölder filtration for A together with Lemma 7.23, A is filtered with quotients of sheaves K_i each of them fits into the short exact sequence

$$0 \to K_i \to \mathcal{E}_{\{x_i\}\times X} \to \mathcal{O}_{Z_i} \to 0$$

in $\mathrm{Coh}(X)$ for some 0-subschemes $Z_i \subset X$. Here $\Gamma^0_{b'}(E) \cong A \in V^{\widehat{\Gamma}}_{\mathcal{B}_{m\ell,b\ell}}(2)$ implies $A \in V^{\widehat{\Upsilon}}_{\mathrm{Coh}(X)}(2,3)$. An easy induction on the number of K_i in A shows that $A \in V^{\widehat{\Upsilon}}_{\mathrm{Coh}(X)}(1,3)$ and so $A \in V^{\widehat{\Upsilon}}_{\mathrm{Coh}(X)}(3)$. Therefore, $Z_i = \emptyset$ for all i and so $\widehat{\Gamma}^2_b\Gamma^0_{b'}(E) \in \mathrm{Coh}^0(X)$. Now consider the convergence of the Spectral Sequence 9.1 for E. We have the short exact sequence

$$0 \to \widehat{\Gamma}^0_b\Gamma^1_{b'}(E) \to \widehat{\Gamma}^2_b\Gamma^0_{b'}(E) \to G \to 0$$

in $\mathcal{B}_{m\ell,b\ell}$, where G is a subobject of E and so $G \in \mathcal{F}'_{m\ell,b\ell}$. Now $\widehat{\Gamma}^2_b\Gamma^0_{b'}(E) \in \mathrm{Coh}^0(X) \subset \mathcal{T}'_{m\ell,b\ell}$ implies $G = 0$ and so $\widehat{\Gamma}^0_b\Gamma^1_{b'}(E) \cong \widehat{\Gamma}^2_b\Gamma^0_{b'}(E)$. Then we have $\Gamma^0_{b'}(E) \cong \Gamma^0_{b'}\widehat{\Gamma}^0_b\Gamma^1_{b'}(E) = 0$. This is not possible as $\Gamma^0_{b'}(E) \neq 0$. Therefore, we have the strict inequality $\Im Z_{m'\ell,b'\ell}(\Gamma^0_{b'}(E)) < 0$ as required.

□

Lemma 9.3. *We have the following:*

(1) if $E \in \mathcal{T}'_{m'\ell,b'\ell}$ then $\widehat{\Gamma}^2_b(E) = 0$,

(2) if $E \in \mathcal{F}'_{m'\ell,b'\ell}$ then $\widehat{\Gamma}^0_b(E) = 0$,

(3) if $E \in \mathcal{T}'_{m\ell,b\ell}$ then $\Gamma^2_{b'}(E) = 0$, and

(4) if $E \in \mathcal{F}'_{m\ell,b\ell}$ then $\Gamma^0_{b'}(E) = 0$.

Proof. First let us prove (1). Let $E \in \mathcal{T}'_{m'\ell,b'\ell}$. From the convergence of the Spectral Sequence 9.1 for E, we have the short exact sequence

$$0 \to Q \to \Gamma^0_{b'}\widehat{\Gamma}^2_b(E) \to \Gamma^2_{b'}\widehat{\Gamma}^1_b(E) \to 0$$

in $\mathcal{B}_{m'\ell,b'\ell}$. Here Q is a quotient of E and so $Q \in \mathcal{T}'_{m'\ell,b'\ell}$. Then $\Gamma^0_{b'}\widehat{\Gamma}^2_b(E)$ fits into the short exact sequence

$$0 \to T \to \Gamma^0_{b'}\widehat{\Gamma}^2_b(E) \to F \to 0$$

in $\mathcal{B}_{m'\ell,b'\ell}$ for some $T \in \mathcal{T}'_{m'\ell,b'\ell}$ and $F \in \mathcal{F}'_{m'\ell,b'\ell}$. Now apply the FM transform $\widehat{\Gamma}$ and consider the long exact sequence of $\mathcal{B}_{m\ell,b\ell}$-cohomologies. Then we have $\widehat{\Gamma}^0_b(T) = 0$, $\widehat{\Gamma}^1_b(T) \cong \widehat{\Gamma}^0_b(F)$. By (2)(ii) of Proposition 9.2, $\Im Z_{m\ell,b\ell}(\widehat{\Gamma}^0_b(F)) \leq 0$ and by (1)(ii) of Proposition 9.2, $\Im Z_{m\ell,b\ell}(\widehat{\Gamma}^2_b(T)) \geq 0$. So $\Im Z_{m\ell,b\ell}(\widehat{\Gamma}(T)) \geq 0$, and by Proposition 5.4, $\Im Z_{m'\ell,b'\ell}(T) \leq 0$. Since $T \in \mathcal{T}'_{m'\ell,b'\ell}$, we have $\Im Z_{m'\ell,b'\ell}(T) = 0$ and $(m'\ell)^2 \operatorname{ch}_1^{b'\ell}(T) = 0$. From Lemma 2.27, $T \cong T_0$ for some $T_0 \in \operatorname{Coh}^0(X)$. But $\operatorname{Coh}^0(X) \subset V^{\widehat{\Gamma}}_{\mathcal{B}_{m\ell,b\ell}}(0)$. Hence, $T = 0$ and so $Q = 0$. Then $\Gamma^0_{b'}\widehat{\Gamma}^2_b(E) \cong \Gamma^2_{b'}\widehat{\Gamma}^1_b(E)$ and so we have $\widehat{\Gamma}^2_b(E) \cong \widehat{\Gamma}^2_b\Gamma^2_{b'}\widehat{\Gamma}^1_b(E) = 0$ as required.

Proofs of (2),(3) and (4) are similar to that of (1). □

Corollary 9.4. *We have the following:*

(1) if $E \in \mathcal{B}_{m'\ell,b'\ell}$ then (i) $\widehat{\Gamma}^2_b(E) \in \mathcal{T}'_{m\ell,b\ell}$, and (ii) $\widehat{\Gamma}^0_b(E) \in \mathcal{F}'_{m\ell,b\ell}$;

(2) if $E \in \mathcal{B}_{m\ell,b\ell}$ then (i) $\Gamma^2_{b'}(E) \in \mathcal{T}'_{m'\ell,b'\ell}$, and (ii) $\Gamma^0_b(E) \in \mathcal{F}'_{m'\ell,b'\ell}$.

Proof. (1) Let $E \in \mathcal{B}_{m'\ell,b'\ell}$. By the definition of torsion theory $\widehat{\Gamma}^2_b(E)$ fits into short exact sequence

$$0 \to T \to \widehat{\Gamma}^2_b(E) \to F \to 0$$

in $\mathcal{B}_{m\ell,b\ell}$ for some $T \in \mathcal{T}'_{m\ell,b\ell}$ and $F \in \mathcal{F}'_{m\ell,b\ell}$. Now apply the FM transform $\widehat{\Gamma}$ and consider the long exact sequence of $\mathcal{B}_{m'\ell,b'\ell}$-cohomologies. Then by Lemma 9.3, $F = 0$ as required.

Similarly one can prove $\widehat{\Gamma}^0_b(E) \in \mathcal{F}'_{m\ell,b\ell}$.

(2) Similar to the proofs in (1).

□

Proposition 9.5. *We have the following:*

(1) if $E \in \mathcal{F}'_{m'\ell,b'\ell}$ then $\widehat{\Gamma}^1_b(E) \in \mathcal{F}'_{m\ell,b\ell}$,

(2) if $E \in \mathcal{T}'_{m'\ell,b'\ell}$ then $\widehat{\Gamma}^1_b(E) \in \mathcal{T}'_{m\ell,b\ell}$,

(3) if $E \in \mathcal{F}'_{m\ell,b\ell}$ then $\Gamma^1_{b'}(E) \in \mathcal{F}'_{m'\ell,b'\ell}$, and

(4) if $E \in \mathcal{T}'_{m\ell,b\ell}$ then $\Gamma^1_{b'}(E) \in \mathcal{T}'_{m'\ell,b'\ell}$.

Proof. Let us prove (1). Let $E \in \mathcal{F}'_{m'\ell,b'\ell}$. By the definition of torsion theory $\widehat{\Gamma}^1_b(E)$ fits into the short exact sequence

$$0 \to T \to \widehat{\Gamma}^1_b(E) \to F \to 0$$

in $\mathcal{B}_{m\ell,b\ell}$ for some $T \in \mathcal{T}'_{m\ell,b\ell}$ and $F \in \mathcal{F}'_{m\ell,b\ell}$. Now we need to show $T = 0$. Apply the FM transform Γ and consider the long exact sequence of $\mathcal{B}_{m'\ell,b'\ell}$-cohomologies. We get $\Gamma^1_{b'}(T) \hookrightarrow \Gamma^1_{b'}\widehat{\Gamma}^1_b(E)$ and $T \in V^{\Gamma}_{\mathcal{B}_{m'\ell,b'\ell}}(1)$. Also by the convergence of the Spectral Sequence 9.1 for E, $\Gamma^1_{b'}\widehat{\Gamma}^1_b(E)$ is a subobject of E. Hence, $\Gamma^1_{b'}(T) \in \mathcal{F}'_{m'\ell,b'\ell}$ implies $\Im Z_{m'\ell,b'\ell}(\Gamma^1_{b'}(T)) \leq 0$. On the other hand, by Proposition 5.4, $\Im Z_{m'\ell,b'\ell}(\Gamma^1_{b'}(T)) = \frac{1}{|\lambda y|^3} \Im Z_{m\ell,b\ell}(T) \geq 0$ as $T \in \mathcal{T}'_{m\ell,b\ell}$. Hence, $\Im Z_{m\ell,b\ell}(T) = 0$ and $T \in \mathcal{T}'_{m\ell,b\ell}$ implies $(m\ell)^2 \operatorname{ch}^{b\ell}_1(T) = 0$. So by Lemma 2.27, $T \cong T_0$ for some $T_0 \in \operatorname{Coh}^0(X)$. Since any object from $\operatorname{Coh}^0(X)$ belongs to $V^{\Gamma}_{\mathcal{B}_{m'\ell,b'\ell}}(0)$, $\Gamma^1_{b'}(T) = 0$. So $T = 0$ as required.

Proofs of (2), (3) and (4) are similar to that of (1).

□

We have the following table of results for the images of $\mathcal{B}$-objects under the FM transforms.

E	$\Gamma^0_{b'}(E)$	$\Gamma^1_{b'}(E)$	$\Gamma^2_{b'}(E)$
$\mathcal{F}'_{m\ell,b\ell}$	0	$\mathcal{F}'_{m'\ell,b'\ell}$	$\mathcal{T}'_{m'\ell,b'\ell}$
$\mathcal{T}'_{m\ell,b\ell}$	$\mathcal{F}'_{m'\ell,b'\ell}$	$\mathcal{T}'_{m'\ell,b'\ell}$	0
E	$\widehat{\Gamma}^0_{b}(E)$	$\widehat{\Gamma}^1_{b}(E)$	$\widehat{\Gamma}^2_{b}(E)$
$\mathcal{F}'_{m'\ell,b'\ell}$	0	$\mathcal{F}'_{m\ell,b\ell}$	$\mathcal{T}'_{m\ell,b\ell}$
$\mathcal{T}'_{m'\ell,b'\ell}$	$\mathcal{F}'_{m\ell,b\ell}$	$\mathcal{T}'_{m\ell,b\ell}$	0

Now we have $\Gamma[1]\left(\mathcal{F}'_{m\ell,b\ell}[1]\right) \subset \mathcal{A}_{m'\ell,b'\ell}$ and $\Gamma[1]\left(\mathcal{T}'_{m\ell,b\ell}\right) \subset \mathcal{A}_{m'\ell,b'\ell}$. Since $\mathcal{A}_{m\ell,b\ell} = \langle \mathcal{F}'_{m\ell,b\ell}[1], \mathcal{T}'_{m\ell,b\ell} \rangle$, $\Gamma[1]\left(\mathcal{A}_{m\ell,b\ell}\right) \subset \mathcal{A}_{m'\ell,b'\ell}$. Similarly, $\widehat{\Gamma}[1]\left(\mathcal{A}_{m'\ell,b'\ell}\right) \subset \mathcal{A}_{m\ell,b\ell}$. The isomorphisms $\widehat{\Gamma}[1] \circ \Gamma[1] \cong \mathrm{id}_{D^b(X)}$ and $\Gamma[1] \circ \widehat{\Gamma}[1] \cong \mathrm{id}_{D^b(X)}$ give us the equivalences

$$\Gamma[1]\left(\mathcal{A}_{m\ell,b\ell}\right) \cong \mathcal{A}_{m'\ell,b'\ell}, \text{ and } \widehat{\Gamma}[1]\left(\mathcal{A}_{m'\ell,b'\ell}\right) \cong \mathcal{A}_{m\ell,b\ell}$$

of the abelian categories as claimed in Theorem 5.5.

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